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RESEARCH PAPER

Acoustomechanical constitutive theory for soft materials

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Abstract Acoustic wave propagation from surrounding medium into a soft material can generate acoustic radiation stress due to acoustic momentum transfer inside the medium and material, as well as at the interface between the two. To analyze acoustic-induced deformation of soft materials, we establish an acoustomechanical constitutive theory by combining the acoustic radiation stress theory and the nonlinear elasticity theory for soft materials. The acoustic radiation stress tensor is formulated by time averaging the momentum equation of particle motion, which is then introduced into the nonlinear elasticity constitutive relation to construct the acoustomechanical constitutive theory for soft materials. Considering a specified case of soft material sheet subjected to two counter-propagating acoustic waves, we demonstrate the nonlinear large deformation of the soft material and analyze the interaction between acoustic waves and material deformation under the conditions of total reflection, acoustic transparency, and acoustic mismatch.

Keywords Acoustomechanical constitutive theory · Acoustic radiation stress · Soft material

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1 Introduction

Acoustic waves carry momentum flux when propagating in a medium, which is capable of generating an acoustic radiation force at the interface of the objects immersed in the acoustic field [1,2]. This force is quite like the optical radiation force produced by electromagnetic waves striking on electrically or magnetically responsive objects. Although the two radiation forces are both attributed to the momentum transfer occurring at the interface between different media, the acoustic radiation force is generally much larger than its optical counterpart [3]. For example, in air at room temperature, the magnitude of the acoustic radiation force is approximately 10^6 times that of its optical counterpart under the same input power. Therefore, the acoustic radiation force may induce large deformation in soft materials. This paper aims to develop an acoustomechanical constitutive theory for nonlinear deformation of soft materials.

The acoustic radiation force is intrinsically a time-averaged force over an oscillation circle of an acoustic wave, namely, a mean residual stress stemming from the nonlinearity of particle momentum. This force is in the form of a stress tensor when exerting on the micro-cubic element of the medium, and is a real force when exerting on an object with an arbitrary surface immersed in the medium. Lord Rayleigh [4,5] first developed a theory of radiation pressure arising from acoustic waves in compressional fluids and obtained the acoustic radiation pressure on a perfectly reflecting surface induced by a normally incident plane acoustic wave in a gas as $(\gamma + 1)\langle E\rangle/2$, where γ is the ratio of specific heat of the gas and $\langle E\rangle$ is the time-averaged energy density of the wave over a circle. Inspired by Lord Rayleigh's pioneering work, numerous studies were carried out to investigate the acoustic radiation force and relevant applications, for

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instance, the study of acoustic radiation pressure on a rigid or compressible sphere [6–9], acoustical trapping and tweezers [10–15], acoustic levitation and contactless handling of matter [16–19], deforming fluid interface and biological tissues [20–22]. This research has demonstrated that the acoustic radiation force generated in the path of traveling acoustic waves is sufficiently large, and can even levitate metallic spheres and deform tissues. Therefore, it is understandable that this acoustic radiation force induces large deformation in soft materials.

Although a great deal of effort has been devoted to studying acoustic radiation forces, there is yet research on large deformation of soft materials induced by these acoustic radiation forces. The present paper aims to establish an acoustomechanical constitutive theory for the nonlinear deformation of soft materials and analyze theoretically the coupling relationship between acoustic waves and material deformation. This work will open the avenue for the field of acoustomechanics for soft materials, which can be employed to investigate the nonlinear deformation and instability behaviors of soft materials. It is also worth pointing out that the present "acoustomechanics" is completely different from the old topic of "acoustoelasticity". Acoustoelasticity focuses on the effects of initial strains or stresses on conventional sound velocities, where the initial deformation is not caused by acoustic waves. In contrast, acoustomechanics concentrates on acoustic-actuated large deformation of soft materials, in which a focused, high-intensity ultrasonic wave can stretch the material to undergo large deformation.

2 Acoustic radiation stress tensor

To formulate an acoustomechanical constitutive theory for soft materials, this section summarizes the fundamental equations of the theory for acoustic radiation stress tensors in an ideal fluid-like material. The theory is expressed in the more familiar Eulerian coordinate system. The motion of a material is governed by a momentum equation [23,24]

$$\rho\left[\frac{\partial \boldsymbol{u}}{\partial t}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\right]+\nabla P=\boldsymbol{0}, \tag{1}$$

and a continuity equation

$$\frac{\partial \rho}{\partial t}+\nabla\cdot(\rho\boldsymbol{u})=0, \tag{2}$$

where ρ is the material density, P is the sound pressure, and $\boldsymbol{u}$ is the velocity field. Combining Eqs. (1) and (2) and introducing the momentum flux tensor $\boldsymbol{T}$, we obtain

$$\frac{\partial(\rho\boldsymbol{u})}{\partial t}+\nabla\cdot\boldsymbol{T}=\boldsymbol{0}, \tag{3}$$

where the momentum flux tensor is given as

$$\boldsymbol{T}=P\boldsymbol{I}+\rho\boldsymbol{u}\otimes\boldsymbol{u}, \tag{4}$$

$\boldsymbol{I}$ being the identity matrix. Following the general custom in fluid mechanics, the momentum flux tensor here represents pressure when its value is positive, while denoting tension when its value is negative. Actually, the momentum flux tensor is the reason of acoustic momentum changing.

Consider an oscillatory motion of the material. For convenience, it is natural to define a mean momentum flux tensor by averaging the pressure tensor over a cycle, denoted by symbol $\langle\cdot\rangle$. Alternatively, this mean momentum flux tensor can be regarded as residual mean stress, which is derived by taking the time average of the momentum equation by noting that $\langle\partial f/\partial t\rangle=0$ when the quantity f is a time period function. Therefore, the mean momentum flux tensor can be expressed as

$$\langle\boldsymbol{T}\rangle=\langle P-P_{\mathrm{a}}\rangle\,\boldsymbol{I}+\langle\rho\boldsymbol{u}\otimes\boldsymbol{u}\rangle\,, \tag{5}$$

where P_{a} is the ambient pressure in an undisturbed state. Since we focus on the fluctuation of excess pressure with respect to the ambient pressure, it will make no difference to use $P-P_{\mathrm{a}}$ instead of P. The second term is the well-known Reynolds stress, standing for the time-averaged transport of acoustic momentum density ρu_i (or ρu_j) with velocity u_j (or u_i) in the x_j (or x_i) direction of the transport. Generally, the Reynolds stress is significant in fluid medium, whereas it is insignificant and can be ignored in solid medium. For solid medium, only in or near resonance, the Reynolds stress plays a dominant role and should be taken into account.

The mean excess pressure $\langle P-P_{\mathrm{a}}\rangle$ is zero at the linear level, but nonzero in general at the nonlinear level, and hence contributes to the change in acoustic momentum. With the ideal fluid-like assumption, the material motion is irrotational. That is to say, the material velocity can be expressed in the form of velocity potential as $\boldsymbol{u}=-\nabla\phi$, ϕ being the fluid velocity potential. The momentum equation can thence be rewritten as

$$\nabla\left[-\frac{\partial\phi}{\partial t}+\frac{1}{2}(\nabla\phi)^2\right]=-\frac{\nabla P}{\rho}. \tag{6}$$

The first law of thermodynamics dictates that $\mathrm{d}h=T\mathrm{d}s+\mathrm{d}p/\rho$, where s and h are the entropy per unit mass and the enthalpy per unit mass of the fluid, and T is the temperature. As the propagation of sound is in general adiabatic so that heat conduction can be ignored, the thermodynamic law simplifies to $\mathrm{d}h=\mathrm{d}P/\rho$ or $\nabla h=\nabla P/\rho$. Integrating Eq. (6) yields the enthalpy per unit mass of the material

$$h=\frac{\partial\phi}{\partial t}-\frac{1}{2}(\nabla\phi)^2+C', \tag{7}$$

where C' is constant in space, but can depend on time. In the nearby region of the undisturbed state, the pressure can be expanded in a Taylor series in terms of enthalpy as

$$P = P_{\mathrm{a}} + \left(\frac{\mathrm{d}p}{\mathrm{d}h}\right)_{\mathrm{s,a}} h + \frac{1}{2}\left(\frac{\mathrm{d}^2 p}{\mathrm{d}h^2}\right)_{\mathrm{s,a}} h^2 + \cdots, \tag{8}$$

where the subscripts "s, a" signify constant entropy and the undisturbed state, respectively. Applying the basic relations $(\mathrm{d}h/\mathrm{d}P)_{\mathrm{s}} = 1/\rho$ and $(\mathrm{d}P/\mathrm{d}\rho)_{\mathrm{s}} = c^2$, we have

$$\left(\frac{\mathrm{d}^2 P}{\mathrm{d}h^2}\right)_{\mathrm{s}} = \left(\frac{\mathrm{d}\rho}{\mathrm{d}h}\right)_{\mathrm{s}} = \left(\frac{\mathrm{d}\rho}{\mathrm{d}P}\right)_{\mathrm{s}}\left(\frac{\mathrm{d}P}{\mathrm{d}h}\right)_{\mathrm{s}} = \frac{\rho}{c^2} = \frac{\rho_{\mathrm{a}}}{c_{\mathrm{a}}^2}. \tag{9}$$

Let all the quantities take undisturbed state values. The pressure can be written as

$$\begin{aligned} P = P_{\mathrm{a}} &+ \rho_{\mathrm{a}}\left[\frac{\partial\phi}{\partial t} - \frac{1}{2}(\nabla\phi)^2 + C'\right] \\ &+ \frac{1}{2}\frac{\rho_{\mathrm{a}}}{c_{\mathrm{a}}^2}\left[\frac{\partial\phi}{\partial t} - \frac{1}{2}(\nabla\phi)^2 + C'\right]^2 + \cdots. \end{aligned} \tag{10}$$

The constant C' is commonly taken as zero in linear acoustics. The enthalpy thus has the form $h = (P - P_{\mathrm{a}})/\rho_{\mathrm{a}} = \partial\phi/\partial t$. Taking time-averaged manipulation of Eq. (10) and keeping terms up to the second order, we obtain

$$\langle P - P_{\mathrm{a}}\rangle = -\frac{1}{2}\rho_{\mathrm{a}}\left\langle(\nabla\phi)^2\right\rangle + \frac{1}{2}\frac{\rho_{\mathrm{a}}}{c_{\mathrm{a}}^2}\left\langle\left(\frac{\partial\phi}{\partial t}\right)^2\right\rangle + C, \tag{11}$$

where $C = \rho_{\mathrm{a}}\langle C'\rangle$ is constant in both space and time. For an open system without rigid boundaries, the value of C can in general be taken as zero. We are, therefore, only interested in truncating up to the second-order terms, as the quadratic terms of ϕ are enough to ensure the accuracy of the mean excess pressure. Substituting Eq. (11) into Eq. (5) yields the acoustic radiation pressure (or stress) tensor [2,25,26]

$$\langle \boldsymbol{T}\rangle = \left[-\frac{1}{2}\rho_{\mathrm{a}}\left\langle(\nabla\phi)^2\right\rangle + \frac{1}{2}\frac{\rho_{\mathrm{a}}}{c_{\mathrm{a}}^2}\left\langle\left(\frac{\partial\phi}{\partial t}\right)^2\right\rangle\right]\boldsymbol{I} + \rho_{\mathrm{a}}\langle \boldsymbol{u}\otimes\boldsymbol{u}\rangle, \tag{12}$$

or equivalently

$$\langle \boldsymbol{T}\rangle = \left(\frac{\langle p^2\rangle}{2\rho_{\mathrm{a}}c_{\mathrm{a}}^2} - \frac{\rho_{\mathrm{a}}\langle \boldsymbol{u}\cdot\boldsymbol{u}\rangle}{2}\right)\boldsymbol{I} + \rho_{\mathrm{a}}\langle \boldsymbol{u}\otimes\boldsymbol{u}\rangle. \tag{13}$$

Upon integration, this stress becomes an actual radiation force

$$\boldsymbol{F}^{\mathrm{rad}} = -\int_{\partial\Omega}\left[\left(\frac{\langle p^2\rangle}{2\rho_{\mathrm{a}}c_{\mathrm{a}}^2} - \frac{\rho_{\mathrm{a}}\langle \boldsymbol{u}\cdot\boldsymbol{u}\rangle}{2}\right)(\boldsymbol{n}\cdot\boldsymbol{I}) + \rho_{\mathrm{a}}\langle(\boldsymbol{n}\cdot\boldsymbol{u})\,\boldsymbol{u}\rangle\right]\mathrm{d}a, \tag{14}$$

where $\boldsymbol{n}$ is the outward positive unit vector on the surface of the selected body.

Finally, given the time-harmonic nature of the present problem, the time-averaged stress is expressed as

$$\langle \boldsymbol{T}\rangle = \frac{1}{2}\mathrm{Re}\left\{\left[\frac{(p^* p)}{2\rho_{\mathrm{a}}c_{\mathrm{a}}^2} - \frac{\rho_{\mathrm{a}}(\boldsymbol{u}^*\cdot\boldsymbol{u})}{2}\right]\boldsymbol{I} + \rho_{\mathrm{a}}\left(\boldsymbol{u}^*\otimes\boldsymbol{u}\right)\right\}, \tag{15}$$

where the superscript asterisk "*" means the complex conjugate of the corresponding variable.

3 Acoustomechanics of soft materials

As formulated in the above section, the acoustical radiation stress scales as $p_0^2/(\rho_{\mathrm{a}}c_{\mathrm{a}}^2)$, and hence the non-dimensional acoustic radiation stress is $p_0^2/(\mu\rho_{\mathrm{a}}c_{\mathrm{a}}^2)$, where p_0 is the amplitude of input sound pressure. Typical acoustic pressure at the focus lies between 0.1 MPa and 4 MPa. Correspondingly, the acoustic radiation stress ranges from 70247 Pa to 112 MPa in air and 4.44 Pa to 7111 Pa in water. Consequently, as the shear modulus of a soft material ranges from dozens of pascals (Pa) to several kilopascals (kPa) [27–31], the acoustic radiation stress is sufficiently large to induce large deformation in the soft material. Compared to conventional mechanical forces, the acoustic radiation stress is comparable in magnitude, but superior due to its non-contact nature and capability of fast manipulation, attractive for a wide range of practical applications.

To facilitate subsequent theoretical formulation, we assume that the soft material is homogeneous, isotropic and nearly incompressible (i.e., det $(\boldsymbol{F}) \approx 1$, $\boldsymbol{F}$ being the deformation gradient tensor), so that its bulk modulus $K = E/[3(1-2v)]$ is much larger than its shear modulus $G = E/[2(1+v)]$. In other words, the soft material behaves like a fluid since its bulk modulus plays a dominant role in wave propagation. Particularly at ultrasonic frequencies, the soft material may be regarded as an ideal fluid medium for wave propagation since it bears negligible dynamical shear stress in such cases, which simplifies tremendously the modeling of wave propagation. However, the assumption of fluid-like material is only for dynamical wave propagation. When static deformation is of concern, the stiffening effect of the soft material needs to be taken into account when large deformation in the material approaches the extension limit.

As previously mentioned, for nonlinear material motion, the acoustic radiation stress is actually a residual mean stress over a period. If the period is small enough, the smooth dynamic process of the stress cannot be identified, which naturally exhibits a steady state of stress, as well as a steady state of the induced deformation. To satisfy this condition, the ultrasonic sound with frequencies beyond 10^6 Hz can be selected as the input sound field, which will generate a dynamic acoustic radiation stress varying in a very short period (less than 10^{-6} s). This stress can, therefore, be regarded as a steady-state averaged stress to cause static material deformation.

Under the above considerations, we next formulate the acoustomechanics of soft materials by combining the nonlinear large deformation mechanics of soft materials and the acoustic radiation stress theory. Since the acoustic radiation stress is a field force, it can be considered an "insider", i.e., as part of material law to establish the acoustomechanics of soft materials. The nonlinear mechanics of soft materials can be summarized as below. We consider a continuum material particle at a particular time as a reference state and represent this material particle using its position $\boldsymbol{X}$. In the current state over a time t, the material particle moves to the new position $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{X}, t)$ with a material deformation gradient $\boldsymbol{F} = \partial \boldsymbol{x}(\boldsymbol{X}, t)/\partial \boldsymbol{X}$. The Cauchy stress is related to the first Piola-Kirchhoff stress as $\boldsymbol{\sigma} = \boldsymbol{s} \cdot \boldsymbol{F}^{\mathrm{T}}/\det(\boldsymbol{F})$. Let $\mathrm{d}V(\boldsymbol{X})$ be a volume element with mass density $\rho(\boldsymbol{X})$ and body force $\hat{\boldsymbol{F}}(\boldsymbol{X}, t)$. Let $\boldsymbol{N}(\boldsymbol{X})\mathrm{d}A(\boldsymbol{X})$ be a surface element with surface force $\hat{\boldsymbol{T}}(\boldsymbol{X}, t)$. $\mathrm{d}A(\boldsymbol{X})$ is the area of the element and $\boldsymbol{N}(\boldsymbol{X})$ is the unit vector normal to the element with positive value towards outside. Force balance of the volume element dictates that $\partial \boldsymbol{s}/\partial \boldsymbol{X} + \hat{\boldsymbol{F}} = \rho \partial^2 \boldsymbol{x}/\partial t^2$, with force boundary condition $\boldsymbol{s} \cdot \boldsymbol{N} = \hat{\boldsymbol{T}}$.

To calculate the acoustic radiation stress in the soft material, sound pressure and velocity fields in and out of the medium need to be determined first. With reference to Fig. 1, consider a thin sheet of soft material impacted by time-harmonic sound field $p(\boldsymbol{x}, t) = p_0 \mathrm{e}^{-\mathrm{j}(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}$, where $\boldsymbol{k}$ is the wavenumber vector. Wave propagation in the material is governed by momentum equation $\nabla \cdot \boldsymbol{\sigma} = \rho \partial^2 \boldsymbol{u}/\partial t^2$ (in Eulerian coordinates). As demonstrated in the Appendix, the pressure and velocity fields can be obtained using the continuity conditions on the boundary interface. Once the pressure and velocity are known, the acoustic radiation stress tensor in and out of the material can be calculated by time-averaging the corresponding variables, as

$$\left\langle \boldsymbol{T}^{\text{outside}} \right\rangle \equiv \left(\frac{\left\langle p_1^2 \right\rangle}{2\rho_1 c_1^2} - \frac{\rho_1 \left\langle \boldsymbol{u}_1 \cdot \boldsymbol{u}_1 \right\rangle}{2} \right) \boldsymbol{I} + \rho_1 \left\langle \boldsymbol{u}_1 \otimes \boldsymbol{u}_1 \right\rangle, \quad (16)$$

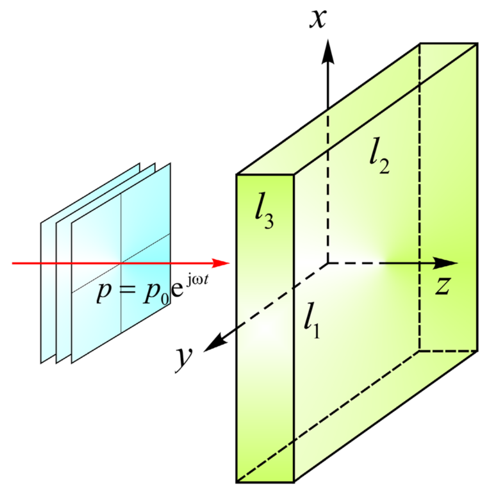

Fig. 1 Illustration of the deformation of a soft material induced by acoustical radiation stress from reference state (L_1, L_2, L_3) to current state (l_1, l_2, l_3)

$$\left\langle \boldsymbol{T}^{\text{inside}} \right\rangle \equiv \left(\frac{\left\langle p_2^2 \right\rangle}{2\rho_2 c_2^2} - \frac{\rho_2 \left\langle \boldsymbol{u}_2 \cdot \boldsymbol{u}_2 \right\rangle}{2} \right) \boldsymbol{I} + \rho_2 \left\langle \boldsymbol{u}_2 \otimes \boldsymbol{u}_2 \right\rangle, \quad (17)$$

where the subscripts "1, 2" represent the outside medium and the inside medium, respectively.

The Cauchy stress in the soft material can be expressed as the sum of elastic-deformation induced mechanical stress and time-averaged acoustic radiation stress

$$\boldsymbol{\sigma} = \frac{\boldsymbol{F}}{\det(\boldsymbol{F})} \frac{\partial W(\boldsymbol{F})}{\partial \boldsymbol{F}} - \langle \boldsymbol{T} \rangle, \quad (18)$$

where $W(\boldsymbol{F})$ is the Helmholtz free energy related to the elastic deformation of the material, which is a symmetric function of the principal stretches ($\lambda_1, \lambda_2, \lambda_3$) for an isotropic material (Fig. 1). Unlike electric or magnetic fields, the input sound field causes no polarization of the soft material, and hence the mechanical stress is decoupled from the acoustical radiation stress, which justifies the simple summation of the two stresses.

For a compressible soft material, if the three-dimensional input sound field in the selected Cartesian coordinates agrees with the principal directions, the Cauchy stress can be expressed in terms of principal stretches as

$$\boldsymbol{\sigma} = \begin{bmatrix} \frac{\lambda_1}{J}\frac{\partial W(\lambda_1,\lambda_2,\lambda_3)}{\partial \lambda_1} - \langle T_{11} \rangle & -\langle T_{12} \rangle & -\langle T_{13} \rangle \\ -\langle T_{21} \rangle & \frac{\lambda_2}{J}\frac{\partial W(\lambda_1,\lambda_2,\lambda_3)}{\partial \lambda_2} - \langle T_{22} \rangle & -\langle T_{23} \rangle \\ -\langle T_{31} \rangle & -\langle T_{32} \rangle & \frac{\lambda_3}{J}\frac{\partial W(\lambda_1,\lambda_2,\lambda_3)}{\partial \lambda_3} - \langle T_{33} \rangle \end{bmatrix}, \quad (19)$$

where $J = \det(\boldsymbol{F})$ is the Jacobian determinant of the deformation gradient. If the input sound field is normally impacting the soft material along one of the principal directions (or Cartesian coordinates), the Cauchy stress becomes

$$\boldsymbol{\sigma} = \begin{bmatrix} \frac{\lambda_1}{J}\frac{\partial W(\lambda_1,\lambda_2,\lambda_3)}{\partial\lambda_1} - \langle T_{11}\rangle & 0 & 0 \\ 0 & \frac{\lambda_2}{J}\frac{\partial W(\lambda_1,\lambda_2,\lambda_3)}{\partial\lambda_2} - \langle T_{22}\rangle & 0 \\ 0 & 0 & \frac{\lambda_3}{J}\frac{\partial W(\lambda_1,\lambda_2,\lambda_3)}{\partial\lambda_3} - \langle T_{33}\rangle \end{bmatrix}. \tag{20}$$

If the soft material is nearly incompressible, $\det(\boldsymbol{F}) \approx 1$. Consider it to be approximately incompressible, thus the Cauchy stress can be expressed as

$$\boldsymbol{\sigma} = \boldsymbol{F}\frac{\partial W(\boldsymbol{F})}{\partial \boldsymbol{F}} - \langle \boldsymbol{T}\rangle - p_{\mathrm{h}}\boldsymbol{I}, \tag{21}$$

where p_{h} is a Lagrange multiplier to satisfy the constraint of near incompressibility (i.e., a yet arbitrary constant scalar to match with the initial and boundary conditions), which is actually the hydrostatic pressure and taken as a constant since the dimensions of the material layer considered in the present study are assumed to be far below the deep where it is located in the fluid medium. We notice that nearly incompressible materials can be approximately modeled by adopting the incompressible model, since it can be numerically demonstrated that the incompressible model of Eq. (21) leads to almost the same results as the compressible model of Eq. (18) when both models are applied to characterize nearly incompressible materials. If the three-dimensional input sound field in the selected Cartesian coordinates agrees with the principal directions, the Cauchy stress can be written in matrix form, as

$$\boldsymbol{\sigma} = \begin{bmatrix} \lambda_1\frac{\partial W(\lambda_1,\lambda_2)}{\partial\lambda_1} - \langle T_{11}\rangle - p_{\mathrm{h}} & -\langle T_{12}\rangle & -\langle T_{13}\rangle \\ -\langle T_{21}\rangle & \lambda_2\frac{\partial W(\lambda_1,\lambda_2)}{\partial\lambda_2} - \langle T_{22}\rangle - p_{\mathrm{h}} & -\langle T_{23}\rangle \\ -\langle T_{31}\rangle & -\langle T_{32}\rangle & -\langle T_{33}\rangle - p_{\mathrm{h}} \end{bmatrix}. \tag{22}$$

If the sound field is normally incident along one of the principal directions (or Cartesian coordinates), the Cauchy stress is simplified as

$$\boldsymbol{\sigma} = \begin{bmatrix} \lambda_1\frac{\partial W(\lambda_1,\lambda_2)}{\partial\lambda_1} - \langle T_{11}\rangle - p_{\mathrm{h}} & 0 & 0 \\ 0 & \lambda_2\frac{\partial W(\lambda_1,\lambda_2)}{\partial\lambda_2} - \langle T_{22}\rangle - p_{\mathrm{h}} & 0 \\ 0 & 0 & -\langle T_{33}\rangle - p_{\mathrm{h}} \end{bmatrix}. \tag{23}$$

Since the acoustic fields should be solved in Eulerian coordinates, they significantly depend upon the deformation of material. In turn, the acoustic fields generate acoustic radiation stress, which induces material deformation in Lagrange coordinates. Bearing in mind this coupling interaction between material deformation and acoustic fields, one can deal with the boundary-value problem by applying an incremental iterative scheme. Starting from the undeformed state, one needs to exert mechanical and acoustic loads with small amplitudes, determine the acoustic field by solving a boundary-value problem over the fixed configuration body, and determine the displacement field by solving another boundary-value problem. Further, one should update the body configuration using the increment of displacement field, and give small increments to both the mechanical and acoustic loads. Repeating the above procedures, one can obtain the finial steady-state deformation until the loads are increased to pre-specified levels and balance the deformation stress.

4 A soft material sheet between two opposing sound inputs

We now employ the acoustomechanical theory to investigate the nonlinear deformation of a thin sheet of soft material subjected to two opposing acoustic fields. As shown in Fig. 2a, the sheet is immersed in an isotropic medium with dimensions (L_1, L_2, L_3) in the undeformed state. When impacted by two acoustic fields having the same amplitude, frequency, and phase position, but opposite propagation directions, the sheet is deformed to dimensions (l_1, l_2, l_3) in the current state. The two input acoustic fields are symmetric with respect to the midplane of soft material sheet, which is convenient to treat the system as one of static deformation because the midplane remains stationary when subjected to two acoustic stresses of equal magnitude, but opposing directions. Let the outside medium and the soft material have acoustic impedance $\rho_1 c_1$ and $\rho_2 c_2$. The Cartesian coordinates (x, y, z) are located on the left side of the soft material, as shown in Fig. 2. Two opposing acoustic fields are incident on the material along the z-direction: the left side field is $p_{\mathrm{L}}(z,t) = p_0 \mathrm{e}^{-\mathrm{j}(k_{1z}z-\omega t)}$ and the right side one is $p_{\mathrm{R}}(z,t) = p_0 \mathrm{e}^{-\mathrm{j}[-k_{1z}(z-l_3)-\omega t]}$, where p_0 is the amplitude, k_{1z} is the wavenumber in the z-direction, and ω is the angular frequency.

Let the thickness of the soft material sheet be thin and comparable to acoustic wavelength $\Lambda = 2\pi\, c_2/\omega$ in the soft material, and let the in-plane dimensions of the sheet be infinitely large. In practice, such conditions may be satisfied by a sheet thickness of 1 mm and in-plane dimensions and spot size of acoustic fields exceeding 10 mm. Following the custom of nonlinear deformation analysis of soft materials, we consider only homogeneous deformation of the soft material with principal stretches (λ_1, λ_2, λ_3).

Since the soft material and the outside medium are both assumed isotropic, the acoustic fields are homogeneously distributed in the two different media. In particular, when the

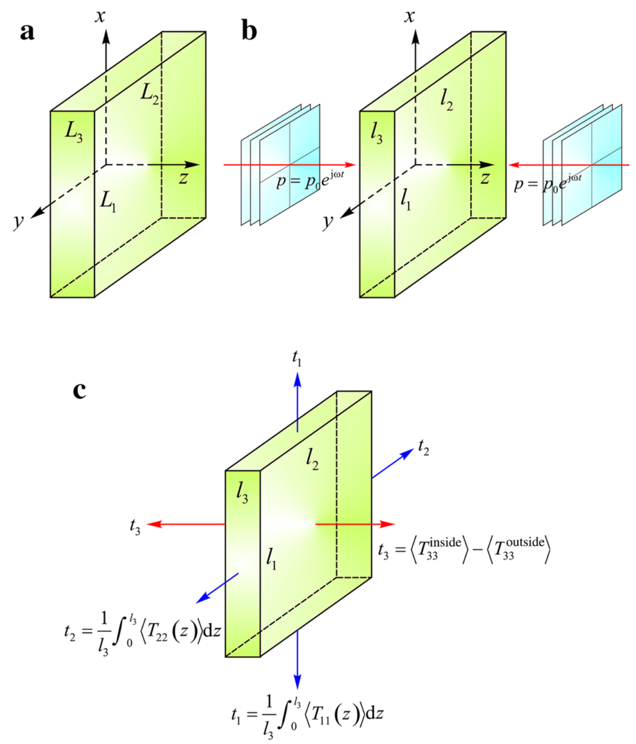

Fig. 2 Deformation of a soft material sheet induced by acoustical radiation stress under two opposing sound pressure. The outside and inside media have acoustic impedance $\rho_1 c_1$ and $\rho_2 c_2$, respectively. **a** In the reference state, the soft material sheet has dimensions (L_1, L_2, L_3). **b** In the current state, the sheet deforms to dimensions (l_1, l_2, l_3) under two opposing acoustic fields $p_L = p_{L0} e^{j\omega t}$ and $p_R = p_{R0} e^{j\omega t}$. **c** Equivalent mechanical stress induced by acoustic radiation pressure

acoustic fields are incident along the z-direction, only the principal acoustic radiation stresses exist

$$\langle T_{11} \rangle = \langle T_{22} \rangle = \frac{1}{2} \mathrm{Re} \left[\frac{(p^* p)}{2 \rho_a c_a^2} \right] - \frac{1}{4} \mathrm{Re} \left[\rho_a \left(u_z^* \cdot u_z \right) \right], \tag{24}$$

$$\langle T_{33} \rangle = \frac{1}{2} \mathrm{Re} \left[\frac{(p^* p)}{2 \rho_a c_a^2} \right] + \frac{1}{4} \mathrm{Re} \left[\rho_a \left(u_z^* \cdot u_z \right) \right]. \tag{25}$$

When the soft material is nearly incompressible ($\det(\boldsymbol{F}) \approx 1$), both the acoustic fields and elastic deformation can be calculated by taking the soft material as a compressible material. However, in the case of near incompressibility, the compressible deformation solution is almost the same as the incompressible deformation solution. Therefore, the assumption of incompressibility is adopted here to model the large deformation behavior of soft materials. In accordance with Eq. (23), the acoustomechanical constitutive theory of a nearly incompressible soft material is given as

$$\sigma_1 - \sigma_3 = \lambda_1 \frac{\partial W(\lambda_1, \lambda_2)}{\partial \lambda_1} - (\langle T_{11} \rangle - \langle T_{33} \rangle), \tag{26}$$

$$\sigma_2 - \sigma_3 = \lambda_2 \frac{\partial W(\lambda_1, \lambda_2)}{\partial \lambda_2} - (\langle T_{22} \rangle - \langle T_{33} \rangle). \tag{27}$$

The Helmholtz free energy function of soft material due to stretching deformation is expressed following the Gent model [32], as

$$W(\boldsymbol{F}) = -\frac{\mu J_{\lim}}{2} \ln \left(1 - \frac{\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3}{J_{\lim}} \right), \tag{28}$$

where μ is the shear modulus and $J_{\lim}$ is the extension limit. When $J_{\lim}$ becomes infinitely large, the Gent model covers the neo-Hookean model. When subjected to symmetric acoustic fields (Fig. 2), the Cauchy stress in the x- and y-directions balance the outside hydrostatic pressure, namely, $\frac{1}{l_3} \int_0^{l_3} \sigma_1 \mathrm{d}z = \frac{1}{l_3} \int_0^{l_3} \sigma_2 \mathrm{d}z = -p_h$. Meanwhile, the Cauchy stress in the z-direction balances the combination of outside acoustic radiation stress and hydrostatic pressure as $\sigma_3 = -\langle T_{33}^{\text{outside}} \rangle - p_h$. Under these force boundary conditions, Eqs. (26) and (27) become

$$t_1 - t_3 = \frac{\mu \left(\lambda_1^2 - \lambda_1^{-2} \lambda_2^{-2} \right)}{1 - \left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3 \right) / J_{\lim}}, \tag{29}$$

$$t_2 - t_3 = \frac{\mu \left(\lambda_2^2 - \lambda_1^{-2} \lambda_2^{-2} \right)}{1 - \left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3 \right) / J_{\lim}}, \tag{30}$$

where

$$t_1 = \frac{1}{l_3} \int_0^{l_3} \langle T_{11}(z) \rangle \mathrm{d}z, \quad t_2 = \frac{1}{l_3} \int_0^{l_3} \langle T_{22}(z) \rangle \mathrm{d}z,$$
$$t_3 = \left\langle T_{33}^{\text{inside}}(l_3) \right\rangle - \left\langle T_{33}^{\text{outside}}(l_3) \right\rangle. \tag{31}$$

As shown in Fig. 2c, Eq. (31) actually gives the equivalent mechanical stresses by homogenizing the acoustic radiation stresses, which are sufficient to deal with the large deformation of the soft material by applying the acoustomechanical constitutive model of Eqs. (29) and (30).

5 Total reflection

Consider the specific case when the material density $\rho_1 \ll \rho_2$ and the sound speed $c_1 \leqslant c_2$, so that the interface is acoustically rigid with respect to the outside surrounding medium. For example, the soft material is coated with metal film and immersed in air. Under such conditions, the incident sound is totally reflected at the interface between the outside medium and the soft material, thus no sound wave penetrates into the latter. Sound pressure and particle velocity in the right side medium of Fig. 2 can, therefore, be written as

$$p_{\mathrm{R}}^{\mathrm{net}} = \mathrm{j}\omega\rho_1 I \mathrm{e}^{\mathrm{j}\omega t}\left[\mathrm{e}^{\mathrm{j}k_{1z}(z-l_3)} + \mathrm{e}^{-\mathrm{j}k_{1z}(z-l_3)}\right], \tag{32}$$

$$u_{\mathrm{R}}^{\mathrm{net}} = \mathrm{j}k_{1z} I \mathrm{e}^{\mathrm{j}\omega t}\left[\mathrm{e}^{-\mathrm{j}k_{1z}(z-l_3)} - \mathrm{e}^{\mathrm{j}k_{1z}(z-l_3)}\right], \tag{33}$$

where I is the amplitude of incident velocity potential. Accordingly, the resultant acoustic radiation stress is $\langle T_{33}^{\mathrm{outside}}(0)\rangle = \langle T_{33}^{\mathrm{outside}}(l_3)\rangle = p_0^2/(\rho_1 c_1^2)$, which exerts a radiation pressure on the surfaces. Since the sound wave does not penetrate into the soft material, this pressure is independent of the thickness of the material and the sound wave polarization.

If we select the initial phase position of the two incident acoustic fields as $\alpha = \pi/2$, the soft material sheet will be stretched by the extension stress $t_3 = p_0^2/(\rho_1 c_1^2)$. For the case of total reflection, Fig. 3 plots the deformation of the soft material induced by acoustic radiation stress. As the material is subjected to a constant extension stress in the z-direction, the non-dimensional incident acoustic field increases monotonously with increasing out-of-plane stretch, and the stretches are independent of the initial sheet thickness. The in-plane stretches are related to the out-of-plane stretch as $\lambda_1 = \lambda_2 = \lambda_3^{-1/2}$.

6 Acoustic transparency

A soft material is acoustically transparent with respect to the outside surrounding medium when its acoustic impedance matches with that of the outside medium (i.e., $\rho_1 c_1 = \rho_2 c_2$). If only one acoustic field is incident on the soft material (Fig. 1), the acoustic wave totally penetrates through it without any reflection, generating acoustic forces $t_1 = t_2 = 0$ and $t_3 = p_0^2/(2\rho_2 c_2^2) - p_0^2/(2\rho_1 c_1^2)$. If two counter-propagating acoustic waves are incident on the soft material as shown in Fig. 2, the real fields are the superposition of the two waves so that the pressure and velocity fields inside and outside the soft material are

$$p_{\mathrm{I}}^{\mathrm{net}} = \mathrm{j}\omega\rho_2 I \frac{c_2}{c_1} \mathrm{e}^{\mathrm{j}\omega t}\left[\mathrm{e}^{-\mathrm{j}k_{2z}z} + \mathrm{e}^{\mathrm{j}k_{2z}(z-l_3)}\right], \tag{34}$$

$$u_{\mathrm{I}}^{\mathrm{net}} = \mathrm{j}k_{2z} I \frac{c_2}{c_1} \mathrm{e}^{\mathrm{j}\omega t}\left[\mathrm{e}^{-\mathrm{j}k_{2z}z} - \mathrm{e}^{\mathrm{j}k_{2z}(z-l_3)}\right], \tag{35}$$

$$p_{\mathrm{R}}^{\mathrm{net}} = \mathrm{j}\omega\rho_1 I \mathrm{e}^{\mathrm{j}\omega t}\left[\mathrm{e}^{\mathrm{j}k_{1z}(z-l_3)} + \mathrm{e}^{-\mathrm{j}(k_{1z}z - k_{1z}l_3 + k_{2z}l_3)}\right], \tag{36}$$

$$u_{\mathrm{R}}^{\mathrm{net}} = \mathrm{j}k_{1z} I \mathrm{e}^{\mathrm{j}\omega t}\left[-\mathrm{e}^{\mathrm{j}k_{1z}(z-l_3)} + \mathrm{e}^{-\mathrm{j}(k_{1z}z - k_{1z}l_3 + k_{2z}l_3)}\right]. \tag{37}$$

Correspondingly, the acoustic radiation stresses are

$$\left\langle T_{11}^{\mathrm{inside}}\right\rangle = \left\langle T_{22}^{\mathrm{inside}}\right\rangle = \frac{p_0^2}{2\rho_2 c_2^2}\left[\mathrm{e}^{\mathrm{j}k_{2z}(2z-l_3)} + \mathrm{e}^{-\mathrm{j}k_{2z}(2z-l_3)}\right], \tag{38}$$

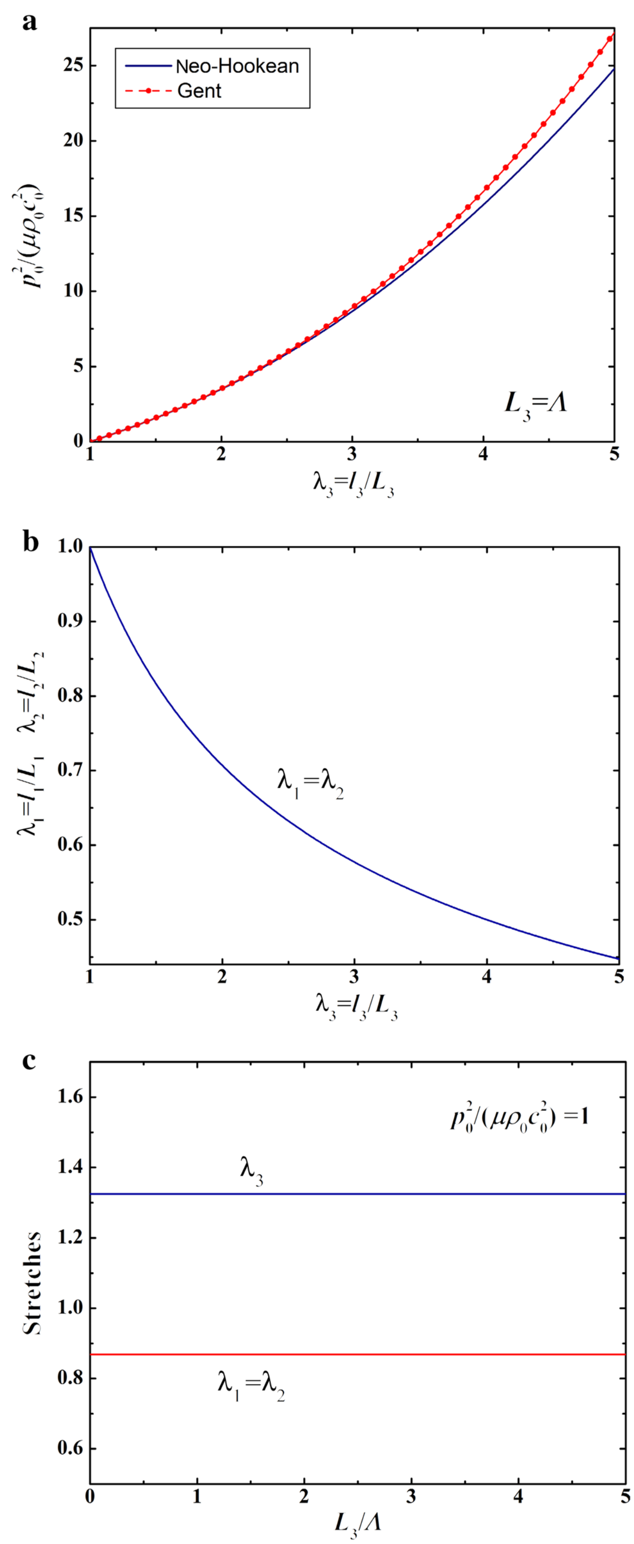

Fig. 3 Deformation of a soft material sheet induced by acoustical radiation stress in the case of total reflection. **a** Out-of-plane stretch plotted as a function of non-dimensional incident acoustic field. **b** Relationship between out-of-plane stretch and in-plane stretches. **c** Stretches of soft material sheets with different initial thicknesses

$$\left\langle T_{33}^{\text{inside}} \right\rangle = \frac{p_0^2}{\rho_2 c_2^2}, \tag{39}$$

$$\left\langle T_{11}^{\text{outside}} \right\rangle = \left\langle T_{22}^{\text{outside}} \right\rangle = \frac{p_0^2}{2\rho_1 c_1^2} \left[\mathrm{e}^{\mathrm{j}(2k_{1z}z - 2k_{1z}l_3 + k_{2z}l_3)} + \mathrm{e}^{-\mathrm{j}(2k_{1z}z - 2k_{1z}l_3 + k_{2z}l_3)} \right], \tag{40}$$

$$\left\langle T_{33}^{\text{outside}} \right\rangle = \frac{p_0^2}{\rho_1 c_1^2}, \tag{41}$$

and the equivalent stresses are

$$t_1 = t_2 = \frac{p_0^2}{\rho_2 c_2^2} \frac{\sin(k_{2z} l_3)}{k_{2z} l_3}, \tag{42}$$

$$t_3 = \frac{p_0^2}{\rho_2 c_2^2} - \frac{p_0^2}{\rho_1 c_1^2}. \tag{43}$$

From Eqs. (42) and (43), it is seen that as the sheet thickness is increased, the equivalent stress t_1 periodically varies its value with a progressively decreasing amplitude around the constant value of zero, approaching eventually this constant value when $k_{2z}l_3 \to \infty$. In contrast, the equivalent stress t_3 is a constant independent of sheet thickness. By substituting Eqs. (42) and (43) into the constitutive equations of (29) and (30), the deformation behavior of the soft material is obtained as shown in Fig. 4. As the equivalent stress t_1 is a periodical function of λ_3 with period $2\pi/(k_{2z}L_3)$, the non-dimensional incident acoustic field varies with λ_3 in a period of $2\pi/(k_{2z}L_3)$, as shown in Fig. 4a. The in-plane stretches are related to the out-of-plane stretch as $\lambda_1 = \lambda_2 = \lambda_3^{-1/2}$ (Fig. 4b). As shown in Fig. 4c, for a given acoustic field input, both the in-plane and out-of-plane stretches are related to the initial sheet thickness L_3. Also, the equivalent stress t_1 is a periodical function of L_3 with period $2\pi/(k_{2z}\lambda_3)$, and hence the stretches alter in a wavy form, asymptotically approaching a constant value when $L_3 \to \infty$.

7 Acoustic mismatch

Generally speaking, the acoustic impedance of a soft material mismatches with that of its outside surrounding medium, with $\rho_1 c_1 \neq \rho_2 c_2$. The acoustic field in each medium consists of both positive- and negative-going waves. In the case of two counter-propagating incident acoustic waves (Fig. 2), the pressure field and the velocity field are separately the superposition of the two opposing fields, as

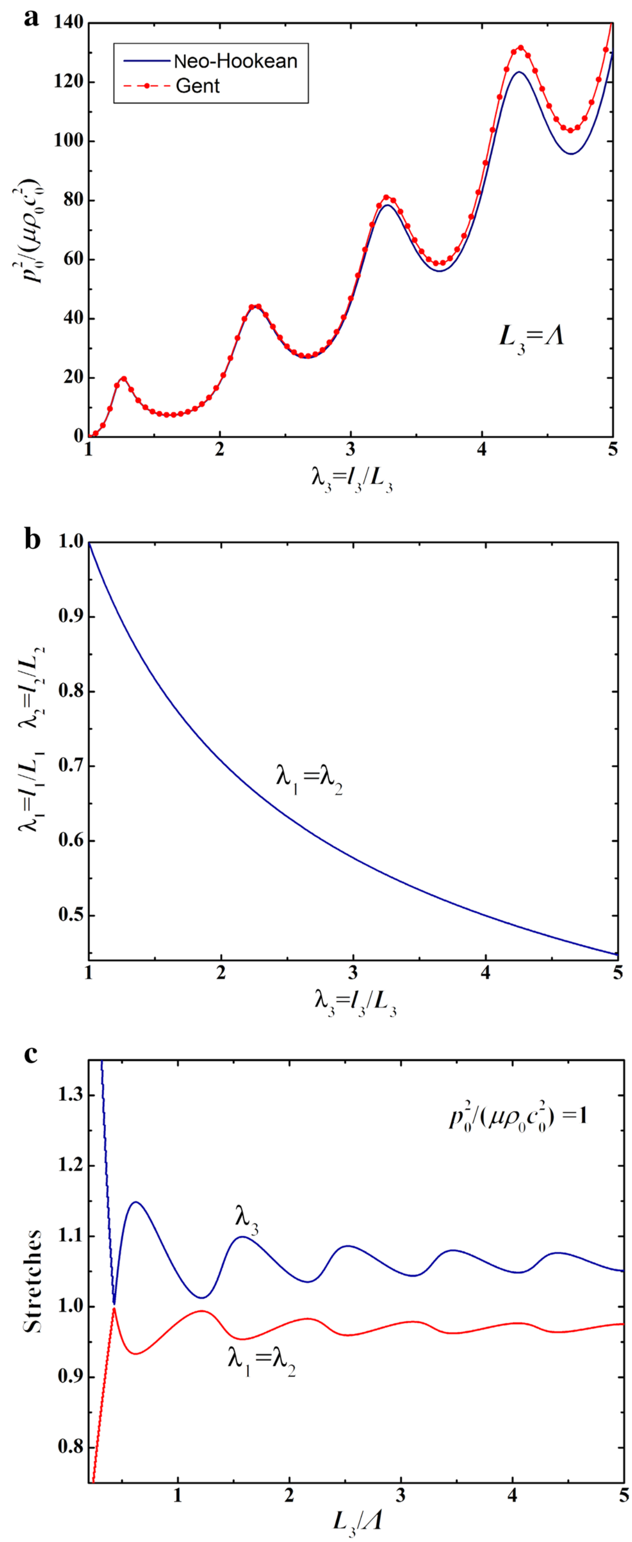

Fig. 4 Deformation of soft material induced by acoustical radiation stress in the case of acoustic transparency. **a** Out-of-plane stretch plotted as a function of non-dimensional incident acoustic field. **b** Relationship between out-of-plane stretch and in-plane stretch. **c** Stretches of soft material sheets with different initial thicknesses

$$p_{\mathrm{I}}^{\mathrm{net}}=\frac{2\omega I\rho_1\rho_2k_{1z}\mathrm{e}^{\mathrm{j}\omega t}\left[\rho_2k_{1z}\sin\left(k_{2z}\left(z-l_3\right)\right)-\rho_2k_{1z}\sin\left(k_{2z}z\right)+\mathrm{j}\rho_1k_{2z}\cos\left(k_{2z}\left(z-l_3\right)\right)+\mathrm{j}\rho_1k_{2z}\cos\left(k_{2z}z\right)\right]}{2\rho_1\rho_2k_{1z}k_{2z}\cos\left(k_{2z}l_3\right)+\mathrm{j}\left(\rho_1^2k_{2z}^2+\rho_2^2k_{1z}^2\right)\sin\left(k_{2z}l_3\right)},\quad(44)$$

$$u_{\mathrm{I}}^{\mathrm{net}}=\frac{2I\rho_1k_{1z}k_{2z}\mathrm{e}^{\mathrm{j}\omega t}\left[\rho_1k_{2z}\sin\left(k_{2z}\left(z-l_3\right)\right)+\rho_1k_{2z}\sin\left(k_{2z}z\right)+\mathrm{j}\rho_2k_{1z}\cos\left(k_{2z}\left(z-l_3\right)\right)-\mathrm{j}\rho_2k_{1z}\cos\left(k_{2z}z\right)\right]}{2\rho_1\rho_2k_{1z}k_{2z}\cos\left(k_{2z}l_3\right)+\mathrm{j}\left(\rho_1^2k_{2z}^2+\rho_2^2k_{1z}^2\right)\sin\left(k_{2z}l_3\right)},\quad(45)$$

$$p_{\mathrm{R}}^{\mathrm{net}}=\mathrm{j}\omega\rho_3I\mathrm{e}^{\mathrm{j}\omega t}\left[\mathrm{e}^{\mathrm{j}k_{3z}(z-l_3)}+\mathrm{e}^{-\mathrm{j}k_{3z}(z-l_3)}\frac{\mathrm{j}\sin\left(k_{2z}l_3\right)\left(\rho_2^2k_{1z}^2-\rho_1^2k_{2z}^2\right)+2\rho_1\rho_2k_{1z}k_{2z}}{2\rho_1\rho_2k_{1z}k_{2z}\cos\left(k_{2z}l_3\right)+\mathrm{j}\left(\rho_1^2k_{2z}^2+\rho_2^2k_{1z}^2\right)\sin\left(k_{2z}l_3\right)}\right],\quad(46)$$

$$u_{\mathrm{R}}^{\mathrm{net}}=\mathrm{j}k_{3z}I\mathrm{e}^{\mathrm{j}\omega t}\left[-\mathrm{e}^{\mathrm{j}k_{3z}(z-l_3)}+\mathrm{e}^{-\mathrm{j}k_{3z}(z-l_3)}\frac{\mathrm{j}\sin\left(k_{2z}l_3\right)\left(\rho_2^2k_{1z}^2-\rho_1^2k_{2z}^2\right)+2I\rho_1\rho_2k_{1z}k_{2z}}{2\rho_1\rho_2k_{1z}k_{2z}\cos\left(k_{2z}l_3\right)+\mathrm{j}\left(\rho_1^2k_{2z}^2+\rho_2^2k_{1z}^2\right)\sin\left(k_{2z}l_3\right)}\right].\quad(47)$$

In such cases, the acoustic radiation stresses are

$$\left\langle T_{11}^{\mathrm{inside}}\right\rangle=\left\langle T_{22}^{\mathrm{inside}}\right\rangle=\frac{1}{2\rho_2c_2^2}\left(A_2B_2^*\mathrm{e}^{-2\mathrm{j}k_{2z}z}+A_2^*B_2\mathrm{e}^{2\mathrm{j}k_{2z}z}\right),\quad(48)$$

$$\left\langle T_{33}^{\mathrm{inside}}\right\rangle=\frac{A_2A_2^*+B_2B_2^*}{2\rho_2c_2^2},\quad(49)$$

$$\left\langle T_{11}^{\mathrm{outside}}\right\rangle=\left\langle T_{22}^{\mathrm{outside}}\right\rangle=\frac{1}{2\rho_1c_1^2}\left(A_1B_1^*\mathrm{e}^{-2\mathrm{j}k_{2z}z}+A_1^*B_1\mathrm{e}^{2\mathrm{j}k_{2z}z}\right),\quad(50)$$

$$\left\langle T_{33}^{\mathrm{outside}}\right\rangle=\frac{A_1A_1^*+B_1B_1^*}{2\rho_1c_1^2},\quad(51)$$

and the equivalent stresses are

$$t_1=t_2=\frac{1}{2\rho_2c_2^2}\left\{\frac{1}{2jk_{2z}l_3}\left[A_2^*B_2\left(\mathrm{e}^{2\mathrm{j}k_{2z}l_3}-1\right)-A_2B_2^*\left(\mathrm{e}^{-2\mathrm{j}k_{2z}l_3}-1\right)\right]\right\},\quad(52)$$

$$t_3=\frac{A_2A_2^*+B_2B_2^*}{2\rho_2c_2^2}-\frac{A_1A_1^*+B_1B_1^*}{2\rho_1c_1^2},\quad(53)$$

where the superscript * means complex conjugates of corresponding variables, and

$$A_1=\mathrm{j}\omega\rho_1I\mathrm{e}^{\mathrm{j}k_{1z}l_3}\times\left[\frac{\mathrm{j}\left(\rho_2^2k_{1z}^2-\rho_1^2k_{2z}^2\right)\sin\left(k_{2z}l_3\right)+2\rho_1\rho_2k_{1z}k_{2z}}{2\rho_1\rho_2k_{1z}k_{2z}\cos\left(k_{2z}l_3\right)+\mathrm{j}\left(\rho_1^2k_{2z}^2+\rho_2^2k_{1z}^2\right)\sin\left(k_{2z}l_3\right)}\right],\quad(54)$$

$$B_1=\mathrm{j}\omega\rho_1I\mathrm{e}^{-\mathrm{j}k_{1z}l_3},\quad(55)$$

$$A_2=\mathrm{j}\omega\rho_1\rho_2k_{1z}I\times\left[\frac{\mathrm{e}^{\mathrm{j}k_{2z}l_3}\left(\rho_2k_{1z}+\rho_1k_{2z}\right)+\left(\rho_1k_{2z}-\rho_2k_{1z}\right)}{2\rho_1\rho_2k_{1z}k_{2z}\cos\left(k_{2z}l_3\right)+\mathrm{j}\left(\rho_1^2k_{2z}^2+\rho_2^2k_{1z}^2\right)\sin\left(k_{2z}l_3\right)}\right],\quad(56)$$

$$B_2=\mathrm{j}\omega\rho_1\rho_2k_{1z}I\times\left[\frac{\mathrm{e}^{-\mathrm{j}k_{2z}l_3}\left(\rho_1k_{2z}-\rho_2k_{1z}\right)+\left(\rho_2k_{1z}+\rho_1k_{2z}\right)}{2\rho_1\rho_2k_{1z}k_{2z}\cos\left(k_{2z}l_3\right)+\mathrm{j}\left(\rho_1^2k_{2z}^2+\rho_2^2k_{1z}^2\right)\sin\left(k_{2z}l_3\right)}\right].\quad(57)$$

To give an intuitive sense of the acoustic radiation stresses, their distributions are plotted in Fig. 5 for two specific cases: current thickness $l_3=\Lambda$ and $l_3=1.5\Lambda$. The in-plane acoustic radiation stress $\langle T_{11}(z)\rangle=\langle T_{22}(z)\rangle$ varies in a sinusoidal way along the z-direction in both the soft material and the outside medium, with significant mismatch at their interface. Only the inside stress deforms the soft material, since we ignore the viscosity of the outside medium. The acoustic radiation stress $\langle T_{33}\rangle$ remains constant along the z-direction both in the soft material and the outside medium, with mismatch at the interface in the $l_3=\Lambda$ case. This difference is actually the equivalent stress t_3 in the z-direction. However, in the $l_3=1.5\Lambda$ case, the inside $\langle T_{33}\rangle$ matches with the outside $\langle T_{33}\rangle$, so that the equivalent stress t_3 vanishes.

Adopting the constitutive equations at the given acoustic radiation stress, the deformation of the soft material can be achieved as shown in Fig. 6. When the current thickness of the sheet is multiples of the half-wavelength of the acoustic waves, the equivalent in-plane and out-of-plane stress t_1 and t_3 indeed exists, but t_1-t_3 approaches zero, which is attributed to the nearly infinite value of the non-dimensional acoustic inputs to attain the deformation. This is also the reason why the stretches periodically approach to one since there is no stress ($t_1-t_3=0$) to induce deformation of soft material. Because of the equal-biaxial extension or compressional nature of the problem, the relationship of the in-plane and out-of-plane stretches $\lambda_1=\lambda_2=\lambda_3^{-1/2}$ always hold true for the normal incident acoustic wave. In principle, whatever direction the acoustic wave is input from, the soft material is stretched when it is relatively softer than outside surrounding medium (i.e., $c_2<c_1$), and is compressed when it is relatively harder than out-

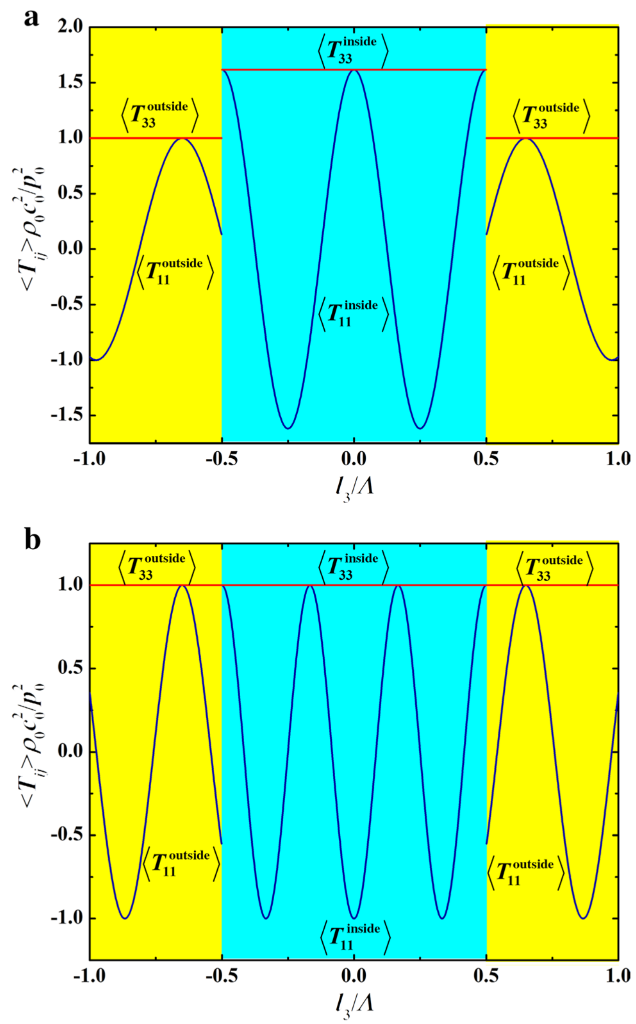

Fig. 5 Distribution of acoustical radiation stress both inside and outside the soft material sheet. **a** $l_3 = \Lambda$. **b** $l_3 = 1.5\Lambda$

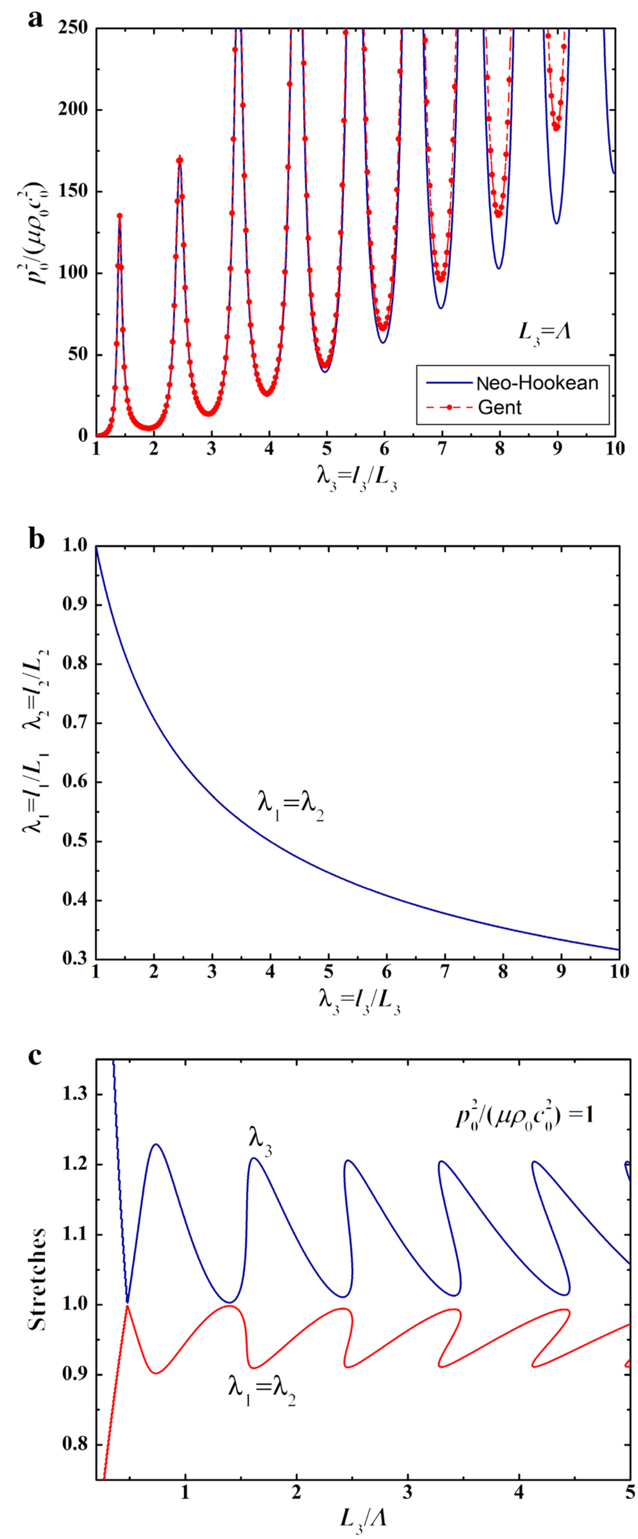

Fig. 6 Deformation of soft material induced by acoustical radiation stress. **a** Dependence of out-of-plane stretch on non-dimensional incident acoustic field. **b** Relationship between out-of-plane stretch and in-plane stretch. **c** Stretches of soft material sheets with different initial thicknesses

sider surrounding medium (i.e., $c_2 > c_1$) when the input acoustic waves have zero phase position at the material interfaces.

8 Acoustic radiation stress boundary condition

The present acoustomechanical constitutive theory is formulated by introducing acoustic radiation stress into the nonlinear stress-stretch relation of soft materials, in which the acoustic radiation stress is homogenized as equivalent mechanical stress to calculate deformation induced in the soft material. This acoustomechanical constitutive theory is generally applicable to soft materials. When an acoustic wave is incident on the soft material, the momentum governing equation of the material dictates that $\nabla \cdot \boldsymbol{T} = -\partial(\rho \boldsymbol{u})/\partial t$. As the acoustic radiation stress deforms the material via the time-averaged residual mean stress, we have $\nabla \cdot \langle \boldsymbol{T} \rangle = \mathbf{0}$ for the acoustic wave considered is time-harmonic and $\langle \partial(\rho \boldsymbol{u})/\partial t \rangle = \mathbf{0}$. According to the theory of nonlinear elasticity, the nonlinear elastic deformation of a nearly incompressible soft material generates stress $\boldsymbol{\sigma}^e = \boldsymbol{F}\partial W(\boldsymbol{F})/\partial \boldsymbol{F} - p_{\mathrm{h}}\boldsymbol{I}$ with material deformation gradient $\boldsymbol{F} = \partial \boldsymbol{x}(\boldsymbol{X}, t)/\partial \boldsymbol{X}$ and material dynamic governing equation $\partial \boldsymbol{s}^e/\partial \boldsymbol{X} + \hat{\boldsymbol{F}} = \rho \partial^2 \boldsymbol{x}/\partial t^2$. The Cauchy stress is related to the first Piola-Kirchhoff stress in the form of $\boldsymbol{\sigma}^e = \boldsymbol{s}^e \cdot \boldsymbol{F}^{\mathrm{T}}/\det(\boldsymbol{F})$. The acoustic radiation stress can be exerted on the surface of soft material as a force boundary condition as $\boldsymbol{\sigma} \cdot \boldsymbol{n} = [\langle \boldsymbol{T}^{\text{inside}} \rangle - \langle \boldsymbol{T}^{\text{outside}} \rangle] \cdot \boldsymbol{n} + \boldsymbol{f}$, where $\boldsymbol{f}$ is the mechanical extension force. For a given deformation state of soft material, the acoustic pressure and velocity fields in the material can be obtained, based on which the acoustic radiation stress can also be determined. Applying the acoustic radiation stress as the force boundary condition and adopting the nonlinear elasticity theory of soft materials, one can favorably solve the acoustomechanical problem of any soft material.

9 Concluding remarks

An acoustomechanical constitutive theory is developed for soft materials by adopting the acoustic radiation stress theory in conjunction with the nonlinear elasticity theory, which demonstrates the giant deformation of soft material under acoustic wave input. The formulation of acoustic radiation stress theory indicates the mean residual stress nature of the acoustic radiation stress by applying time-averaged manipulation over the governing equations. Acoustomechanical stress versus stretch relations for both compressible and nearly incompressible soft materials are given in consideration of the acoustic radiation stress. For the specialized case for two counter-propagating acoustic waves incident simultaneously on a nearly incompressible soft material, the general acoustomechanical constitutive model is specified via a stress homogenization method. The deformation behavior of and stress distribution in a thin sheet of soft material are analyzed for particular cases, including total reflection, acoustic transparency, and acoustic mismatch.

The acoustomechanical behavior of a structure made of soft material is significantly dependent upon its initial and current geometries, which opens a pathway to design novel multifunctional soft devices in a wide range of fields such as robotics, medicine, and biology. The large deformation of soft material induced by acoustic waves enables different stimuli transduction in e.g., mechanical stress, electric fields, and acoustic fields. Also, controlling acoustic wave propagation via an acoustic wave itself is possible, since the acoustic wave causes deformation of the soft material and changes its geometry. As a future prospect, the time-dependent viscoelasticity behavior of soft materials should be considered in acoustomechanical modeling since most soft materials exhibit time-dependent behaviors.

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Appendix: acoustic wave propagation

With reference to Fig. 1, consider a time-harmonic acoustic wave $p(\boldsymbol{x}, t) = p_0 \mathrm{e}^{-\mathrm{j}(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t)}$ with wavenumber vector $\boldsymbol{k}$ incident on the surface of a soft material sheet from its outside surrounding medium. Generally, the incident acoustic wave will generate reflection and transmission not only on the inlet surface, but also on the outlet surface and, in Eulerian coordinates, wave propagation in the medium is governed by dynamical equation $\nabla \cdot \boldsymbol{\sigma} = \rho \partial^2 \boldsymbol{u}/\partial t^2$. For ideal fluid-like materials, this equation degrades to the Helmholtz equation $\rho \partial^2 \boldsymbol{u}/\partial t^2 + \nabla p = \mathbf{0}$. The velocity potential field induced by the incident acoustic wave can be written as

$$\phi_1(\boldsymbol{x}, t) = I \mathrm{e}^{-\mathrm{j}\left(\boldsymbol{k}_1^+ \cdot \boldsymbol{x} - \omega t\right)} + \beta \mathrm{e}^{-\mathrm{j}\left(\boldsymbol{k}_1^- \cdot \boldsymbol{x} - \omega t\right)}, \tag{A1}$$

$$\phi_2(\boldsymbol{x}, t) = \varepsilon \mathrm{e}^{-\mathrm{j}\left(\boldsymbol{k}_2^+ \cdot \boldsymbol{x} - \omega t\right)} + \zeta \mathrm{e}^{-\mathrm{j}\left(\boldsymbol{k}_2^- \cdot \boldsymbol{x} - \omega t\right)}, \tag{A2}$$

$$\phi_3(\boldsymbol{x}, t) = \xi \mathrm{e}^{-\mathrm{j}\left(\boldsymbol{k}_3^+ \cdot \boldsymbol{x} - \omega t\right)}, \tag{A3}$$

where the subscripts "1, 2, and 3" denote the left medium, the soft material sheet, and the right medium, while the superscripts "+" and "-" correspond to positive- and negative-going waves, respectively. The corresponding velocity field and pressure field can be expressed as

$$\boldsymbol{u}_i = -\nabla \phi_i, \quad p_i = \rho_i \frac{\partial \phi_i}{\partial t}, \; i = 1, 2, 3. \tag{A4}$$

Continuity of velocity and acoustic pressure requires

$$u_{1z} = u_{2z}|_{z=0}, \quad u_{2z} = u_{3z}|_{z=l_3}, \tag{A5}$$

$$p_1 = p_2|_{z=0}, \quad p_2 = p_3|_{z=l_3}, \tag{A6}$$

which can be rewritten in the form

$$\begin{aligned} &\mathrm{j}k_{1z} I \mathrm{e}^{-\mathrm{j}(k_{1x}x + k_{1y}y - \omega t)} - \mathrm{j}k_{1z}\beta \mathrm{e}^{-\mathrm{j}(k_{1x}x + k_{1y}y - \omega t)} \\ &\quad = \mathrm{j}k_{2z}\varepsilon \mathrm{e}^{-\mathrm{j}(k_{2x}x + k_{2y}y - \omega t)} - \mathrm{j}k_{2z}\zeta \mathrm{e}^{-\mathrm{j}(k_{2x}x + k_{2y}y - \omega t)}, \end{aligned} \tag{A7}$$

$$\begin{aligned} &\mathrm{j}k_{2z}\varepsilon \mathrm{e}^{-\mathrm{j}(k_{2x}x + k_{2y}y + k_{2z}l_3 - \omega t)} - \mathrm{j}k_{2z}\zeta \mathrm{e}^{-\mathrm{j}(k_{2x}x + k_{2y}y - k_{2z}l_3 - \omega t)} \\ &\quad = \mathrm{j}k_{3z}\xi \mathrm{e}^{-\mathrm{j}(k_{3x}x + k_{3y}y + k_{3z}l_3 - \omega t)}, \end{aligned} \tag{A8}$$

$$\mathrm{j}\omega\rho_1\left[I\mathrm{e}^{-\mathrm{j}(k_{1x}x+k_{1y}y-\omega t)}+\beta\mathrm{e}^{-\mathrm{j}(k_{1x}x+k_{1y}y-\omega t)}\right]=\mathrm{j}\omega\rho_2\left[\varepsilon\mathrm{e}^{-\mathrm{j}(k_{2x}x+k_{2y}y-\omega t)}+\zeta\mathrm{e}^{-\mathrm{j}(k_{2x}x+k_{2y}y-\omega t)}\right], \tag{A9}$$

$$\mathrm{j}\omega\rho_2\left[\varepsilon\mathrm{e}^{-\mathrm{j}(k_{2x}x+k_{2y}y+k_{2z}l_3-\omega t)}+\zeta\mathrm{e}^{-\mathrm{j}(k_{2x}x+k_{2y}y-k_{2z}l_3-\omega t)}\right]=\mathrm{j}\omega\rho_3\left[\xi\mathrm{e}^{-\mathrm{j}(k_{3x}x+k_{3y}y+k_{3z}l_3-\omega t)}\right]. \tag{A10}$$

If the same medium occupies the left side and the right side, i.e., $\rho_1=\rho_3$ and $k_1=k_3$, one has

$$\beta=\frac{I\mathrm{e}^{\mathrm{j}k_{2z}l_3}\left(\rho_2^2k_{1z}^2-\rho_1^2k_{2z}^2\right)+I\mathrm{e}^{-\mathrm{j}k_{2z}l_3}\left(\rho_1^2k_{2z}^2-\rho_2^2k_{1z}^2\right)}{\mathrm{e}^{\mathrm{j}k_{2z}l_3}\left(\rho_1k_{2z}+\rho_2k_{1z}\right)^2-\mathrm{e}^{-\mathrm{j}k_{2z}l_3}\left(\rho_1k_{2z}-\rho_2k_{1z}\right)^2}, \tag{A11}$$

$$\varepsilon=\frac{2I\rho_1k_{1z}\mathrm{e}^{\mathrm{j}k_{2z}l_3}\left(\rho_2k_{1z}+\rho_1k_{2z}\right)}{\mathrm{e}^{\mathrm{j}k_{2z}l_3}\left(\rho_1k_{2z}+\rho_2k_{1z}\right)^2-\mathrm{e}^{-\mathrm{j}k_{2z}l_3}\left(\rho_1k_{2z}-\rho_2k_{1z}\right)^2}, \tag{A12}$$

$$\zeta=\frac{2I\rho_1k_{1z}\mathrm{e}^{-\mathrm{j}k_{2z}l_3}\left(\rho_1k_{2z}-\rho_2k_{1z}\right)}{\mathrm{e}^{\mathrm{j}k_{2z}l_3}\left(\rho_1k_{2z}+\rho_2k_{1z}\right)^2-\mathrm{e}^{-\mathrm{j}k_{2z}l_3}\left(\rho_1k_{2z}-\rho_2k_{1z}\right)^2}, \tag{A13}$$

$$\xi=\frac{4I\rho_1\rho_2k_{1z}k_{2z}\mathrm{e}^{\mathrm{j}k_{1z}l_3}}{\mathrm{e}^{\mathrm{j}k_{2z}l_3}\left(\rho_1k_{2z}+\rho_2k_{1z}\right)^2-\mathrm{e}^{-\mathrm{j}k_{2z}l_3}\left(\rho_1k_{2z}-\rho_2k_{1z}\right)^2}. \tag{A14}$$

When two counter-propagating acoustic waves are normally incident on a soft material sheet (Fig. 2), the acoustic pressure and velocity can be expressed as

$$p_1^{\mathrm{net}}=\mathrm{j}\omega\rho_1I\mathrm{e}^{\mathrm{j}\omega t}\left[\mathrm{e}^{-\mathrm{j}k_{1z}z}+\mathrm{e}^{\mathrm{j}k_{1z}z}\frac{\mathrm{j}\sin\left(k_{2z}l_3\right)\left(\rho_2^2k_{1z}^2-\rho_1^2k_{2z}^2\right)+2\rho_1\rho_2k_{1z}k_{2z}}{2\rho_1\rho_2k_{1z}k_{2z}\cos\left(k_{2z}l_3\right)+\mathrm{j}\left(\rho_1^2k_{2z}^2+\rho_2^2k_{1z}^2\right)\sin\left(k_{2z}l_3\right)}\right], \tag{A15}$$

$$u_{1z}^{\mathrm{net}}=\mathrm{j}k_{1z}I\mathrm{e}^{\mathrm{j}\omega t}\left[\mathrm{e}^{-\mathrm{j}k_{1z}z}-\mathrm{e}^{\mathrm{j}k_{1z}z}\frac{\mathrm{j}\sin\left(k_{2z}l_3\right)\left(\rho_2^2k_{1z}^2-\rho_1^2k_{2z}^2\right)+2\rho_1\rho_2k_{1z}k_{2z}}{2\rho_1\rho_2k_{1z}k_{2z}\cos\left(k_{2z}l_3\right)+\mathrm{j}\left(\rho_1^2k_{2z}^2+\rho_2^2k_{1z}^2\right)\sin\left(k_{2z}l_3\right)}\right], \tag{A16}$$

$$p_2^{\mathrm{net}}=\frac{2\omega I\rho_1\rho_2k_{1z}\mathrm{e}^{\mathrm{j}\omega t}\left[\rho_2k_{1z}\sin\left(k_{2z}\left(z-l_3\right)\right)-\rho_2k_{1z}\sin\left(k_{2z}z\right)+\mathrm{j}\rho_1k_{2z}\cos\left(k_{2z}\left(z-l_3\right)\right)+\mathrm{j}\rho_1k_{2z}\cos\left(k_{2z}z\right)\right]}{2\rho_1\rho_2k_{1z}k_{2z}\cos\left(k_{2z}l_3\right)+\mathrm{j}\left(\rho_1^2k_{2z}^2+\rho_2^2k_{1z}^2\right)\sin\left(k_{2z}l_3\right)}, \tag{A17}$$

$$u_{2z}^{\mathrm{net}}=\frac{2I\rho_1k_{1z}k_{2z}\mathrm{e}^{\mathrm{j}\omega t}\left[\rho_1k_{2z}\sin\left(k_{2z}\left(z-l_3\right)\right)+\rho_1k_{2z}\sin\left(k_{2z}z\right)+\mathrm{j}\rho_2k_{1z}\cos\left(k_{2z}\left(z-l_3\right)\right)-\mathrm{j}\rho_2k_{1z}\cos\left(k_{2z}z\right)\right]}{2\rho_1\rho_2k_{1z}k_{2z}\cos\left(k_{2z}l_3\right)+\mathrm{j}\left(\rho_1^2k_{2z}^2+\rho_2^2k_{1z}^2\right)\sin\left(k_{2z}l_3\right)}, \tag{A18}$$

$$p_3^{\mathrm{net}}=\mathrm{j}\omega\rho_1I\mathrm{e}^{\mathrm{j}\omega t}\left[\mathrm{e}^{\mathrm{j}k_{1z}(z-l_3)}+\mathrm{e}^{-\mathrm{j}k_{1z}(z-l_3)}\frac{\mathrm{j}\sin\left(k_{2z}l_3\right)\left(\rho_2^2k_{1z}^2-\rho_1^2k_{2z}^2\right)+2\rho_1\rho_2k_{1z}k_{2z}}{2\rho_1\rho_2k_{1z}k_{2z}\cos\left(k_{2z}l_3\right)+\mathrm{j}\left(\rho_1^2k_{2z}^2+\rho_2^2k_{1z}^2\right)\sin\left(k_{2z}l_3\right)}\right], \tag{A19}$$

$$u_{3z}^{\mathrm{net}}=\mathrm{j}k_{1z}I\mathrm{e}^{\mathrm{j}\omega t}\left[-\mathrm{e}^{\mathrm{j}k_{1z}(z-l_3)}+\mathrm{e}^{-\mathrm{j}k_{1z}(z-l_3)}\frac{\mathrm{j}\sin\left(k_{2z}l_3\right)\left(\rho_2^2k_{1z}^2-\rho_1^2k_{2z}^2\right)+2I\rho_1\rho_2k_{1z}k_{2z}}{2\rho_1\rho_2k_{1z}k_{2z}\cos\left(k_{2z}l_3\right)+\mathrm{j}\left(\rho_1^2k_{2z}^2+\rho_2^2k_{1z}^2\right)\sin\left(k_{2z}l_3\right)}\right]. \tag{A20}$$

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