Cut-off frequencies of Lamb waves in various functionally graded thin films

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Cut-off frequencies of Lamb waves in various functionally graded thin films

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An analytical study is carried out on the cut-off frequencies of Lamb waves in freestanding thin films made of various functionally graded elastic, piezoelectric, or piezoelectric–piezomagnetic materials. Results show that the set of cut-off frequencies is a union of two series of approximate arithmetic progression, in which the differences are inversely proportional to the definite integral of a function of the material parameters along thickness. Given the simple and universal relationship between cut-off frequencies and material parameters, this study provides theoretical guidance not only for nondestructive evaluation in engineering applications but for designing high-performance sensors based on Lamb waves. © 2011 American Institute of Physics. [doi:10.1063/1.3640478]

Since Sir Lamb published the detailed treatise on Lamb waves in 1917,[1] Lamb waves have been widely used in engineering applications, such as nondestructive evaluations (NDEs), transducers, resonators, sensors, and so on, because of their high sensitivity and low attenuation.[2,3] Many investigators have explored a number of propagation characteristics in various thin films made of different homogenous materials, such as elastic,[2] piezoelectric,[4,5] and piezoelectric–piezomagnetic materials.[6] Such characteristics include the dispersion properties, wave structures, and cut-off frequencies of Lamb waves.

As a kind of inhomogeneous material, functionally graded materials (FGMs) have recently been proposed to solve problems in the thermal protection systems of aerospace structures, thereby drawing more interest from various engineering fields.[7] With the development of material technology, the concept of FGM has been used not only for common elastic materials but also for smart materials, including piezoelectric and piezoelectric–piezomagnetic materials. Demands from ultrasonic technology and NDE fields have driven research on the wave propagation characteristics in such structures. Although many reports focus on the propagation characteristics of Lamb waves in various inhomogeneous plates,[8–10] most of them primarily discuss dispersion properties and wave structures. Few studies regarding the cut-off frequencies of Lamb waves in inhomogeneous plates have been published.[9,11] Regardless of whether an analytical[9] or numerical method[8] is used, complex mathematical calculations are necessary to obtain numerical results for wave propagation characteristics in certain inhomogeneous structures. This complexity in analysis has always been an obstacle to the practical application of the results.

Given that various smart materials, including piezoelectric and piezoelectric–piezomagnetic materials, are widely used in micro-electro-mechanical systems, many studies have focused on wave propagation behaviors in structures made of such materials. As an important wave propagation characteristic, cut-off frequencies can be used not only for corrosion detection and thickness measurement in different structures but also for various sensors and transducers.[12,13] The cut-off frequencies can be calculated by examining the limiting condition $k \to 0$.[2] Consider that Lamb waves propagate in a thin film made of inhomogeneous materials, such as FGMs, functionally graded piezoelectric materials (FGPMs), and functionally graded piezoelectric–piezomagnetic materials (FGPPMs), and that the thickness of the thin film is h. These are transverse isotropic materials; the xoy coordinate plane is an isotropic plane, the polling direction of the FGPM and FGPPM is the same as the positive direction of the z-axis, and Lamb waves propagate along the positive direction of the x-axis (Fig. 1). Thus, the displacement vector is given by $u = u(x, z, t)$, $v = 0$, $w = w(x, z, t)$. The electrical and the magnetic potential is $\varphi = \varphi(x, z, t)$ and $\psi = \psi(x, z, t)$, respectively.

We first discuss the most complex problem, i.e., Lamb wave propagation in an FGPPM thin film. For brevity, $\exp[ik(x - ct)]$ is omitted; k is the wave number, c is the phase velocity, and $i = \sqrt{-1}$. Considering the limiting condition $k \to 0$, the governing equations are as follows:

$$(c_{44}u')' + \rho\omega^2 u = 0, \tag{1}$$

$$(c_{33}w' + e_{33}\varphi' + f_{33}\psi')' + \rho\omega^2 w = 0, \tag{2}$$

$$(\varepsilon_{33}\varphi' - e_{33}w' + g_{33}\psi')' = 0, \tag{3}$$

$$(\mu_{33}\psi' + g_{33}\varphi' - f_{33}w')' = 0, \tag{4}$$

where c_{33} and c_{44}, e_{33}, f_{33}, ε_{33}, μ_{33}, and g_{33} are the elastic, piezoelectric, piezomagnetic, dielectric, magnetic, and magnetoelectric coefficients, respectively, and ρ is the density of the FGPPM. These are functions of thickness. The symbol "′" represents the first differential with respect to z.

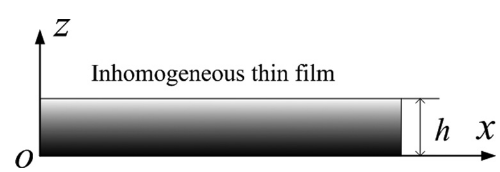

FIG. 1. FGM, FGPM, or FGPPM thin films and Cartesian coordinates.

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Furthermore, the governing equations of the electrical and magnetic potentials in air are

$$\varphi_j^{''} = 0, \quad \psi_j^{''} = 0, \tag{5}$$

where $j = 1, 2$ denotes the physical variable of the upper and bottom half-spaces in air, respectively.

Stresses σ_z and σ_{xz}, electrical displacement D_z, and magnetic induction B_z are shown as follows:

$$\sigma_z = c_{33}w' + e_{33}\varphi' + f_{33}\psi', \quad \sigma_{xz} = c_{44}u', \tag{6}$$

$$D_z = e_{33}w' - \varepsilon_{33}\varphi' - f_{33}\psi', \quad B_z = f_{33}w' - g_{33}\varphi' - \mu_{33}\psi'. \tag{7}$$

The boundary conditions must be satisfied. The traction-free condition at $z = 0, h$ is

$$\sigma_z = 0, \quad \sigma_{xz} = 0. \tag{8}$$

We discuss only the electrical and magnetic open cases,

1. At $z = 0$, $\varphi = \varphi_1$, $D_z = D_{z1}$, $\psi = \psi_1$, $B_z = B_{z1}$,
2. At $z = h$, $\varphi = \varphi_2$, $D_z = D_{z2}$, $\psi = \psi_2$, $B_z = B_{z2}$.

Moreover, considering attenuation tendency at infinity, we obtain the solution of Eq. (5) as $\varphi_j = C_{1j}, \psi_j = C_{2j}$, where C_{1j} and C_{2j}, $j = 1, 2$ are undetermined constants. Considering these solutions, Eq. (7), and electrical and magnetic open boundary condition, we can rewrite Eqs. (3) and (4) as

$$\varepsilon_{33}\varphi^{'} - e_{33}w^{'} + g_{33}\psi^{'} = 0, \quad \mu_{33}\psi^{'} + g_{33}\varphi^{'} - f_{33}w^{'} = 0. \tag{9}$$

Substituting Eq. (9) into Eq. (2) yields

$$(c_E w')^{'} + \rho\omega^2 w = 0, \tag{10}$$

where c_E is an equivalent elastic coefficient and satisfies

$$c_E = c_{33} + e_{33}\frac{e_{33}\mu_{33} - f_{33}g_{33}}{\varepsilon_{33}\mu_{33} - g_{33}^2} + f_{33}\frac{f_{33}\varepsilon_{33} - e_{33}g_{33}}{\varepsilon_{33}\mu_{33} - g_{33}^2}.$$

Similarly, we obtain that $c_E = c_{33}$ in an FGM film and $c_E = c_{33} + e_{33}^2/\varepsilon_{33}$ in an FGPM film.

To solve Eqs. (1) and (10), we use the Wentzel–Kramers–Brillouin (WKB) method and obtain the solutions,[10,11] expressed as

$$u = (\rho c_{44})^{-1/4}\left[a_1\cos\left(\int_0^z \sqrt{\rho/c_{44}}dz\omega_n\right) + a_2\sin\left(\int_0^z \sqrt{\rho/c_{44}}dz\omega_n\right)\right], \tag{11}$$

$$w = (\rho c_E)^{1/4}\left[b_1\cos\left(\int_0^z \sqrt{\rho/c_E}dz\omega_n\right) + b_2\sin\left(\int_0^z \sqrt{\rho/c_E}dz\omega_n\right)\right]. \tag{12}$$

By substituting Eqs. (11) and (12) into traction-free conditions, we obtain a set of homogeneous linear algebraic equations for determining a_1, a_2, b_1, and b_2. Considering sufficient and necessary condition for the existence of a nontrivial solution, we have

$$\tan\left(\int_0^h \sqrt{\rho/c_{44}}dz\omega_{nT}\right) = \frac{A(0)A'(h)\sqrt{\rho_0/c_{44_0}} - A'(0)A(h)\sqrt{\rho_h/c_{44_h}}}{A'(0)A'(h) + A(0)A(h)\omega_{nT}^2\sqrt{\rho_0\rho_h/c_{44_0}c_{44_h}}}\omega_{nT}, \tag{13}$$

$$\tan\left(\int_0^h \sqrt{\rho/c_E}dz\omega_{nL}\right) = \frac{B(0)B'(h)\sqrt{\rho_0/c_{E_0}} - B'(0)B(h)\sqrt{\rho_h/c_{E_h}}}{B'(0)B'(h) + B(0)B(h)\omega_{nL}^2\sqrt{\rho_0\rho_h/c_{E_0}c_{E_h}}}\omega_{nL}, \tag{14}$$

where $A(z) = (\rho c_{44})^{-1/4}$ and $B(z) = (\rho c_E)^{-1/4}$, the subscripts 0 and h represent the parameter at the bottom and upper surfaces of the film, respectively, and ω_{nT} and ω_{nL} are cut-off frequencies ω_n that satisfy Eqs. (11) and (12), respectively.

Observing Eq. (13) when the material properties slowly vary along the thickness and ω_{nT} and ω_{nL} are large parameters, the right side of the equation approaches zero. Hence, we have $\int_0^h \sqrt{\rho/c_{44}}\omega_{nT} \to n\pi$; specifically, $\omega_{nT} \to n\pi/\int_0^h \sqrt{\rho/c_{44}}$. Similarly, we can deduce that $\int_0^h \sqrt{\rho/c_E}\omega_{nL} \to n\pi$ from Eq. (14); specifically, $\omega_{nL} \to n\pi/\int_0^h \sqrt{\rho/c_E}$. Consequently, we have

$$D_n(\omega_T) = \omega_{(n+1)T} - \omega_{nT} \to \pi\bigg/\int_0^h \sqrt{\rho/c_{44}}dz,$$
$$D_n(\omega_L) = \omega_{(n+1)L} - \omega_{nL} \to \pi\bigg/\int_0^h \sqrt{\rho/c_E}dz. \tag{15}$$

Thus, the series of ω_{nT} and ω_{nL} can be considered two approximate arithmetic progressions, in which the difference is the ratio of π to the definite integral of $\sqrt{\rho/c_{44}}$ and $\sqrt{\rho/c_E}$ along the thickness, respectively, as shown in Eq. (1).

From Eqs. (13) and (14), we deduce the cut-off frequencies in a homogenous thin film. The results agree with the

TABLE I. Cut-off frequencies of Lamb waves $\omega_{nT}h$ and $\omega_{nL}h$ (Hzm) in different thin films.

	Homogenous		Type 1		Type 2	Type 3	Type 4	Type 5	Type 6	Type 7	
n	$\omega_{nT}h$	$\omega_{nL}h$	$\omega_{nT}h$	$\omega_{nL}h$	$\omega_{nL}h$	$\omega_{nL}h$	$\omega_{nL}h$	$\omega_{nL}h$	$\omega_{nL}h$	$\omega_{nT}h$	$\omega_{nL}h$
1	7947.67	17344.94	8330.46	18142.85	17418.69	17355.22	17312.59	17340.49	17344.84	7601.36	16564.45
2	15895.34	34689.88	16655.66	36275.20	34839.42	34710.73	34626.81	34681.21	34689.68	15171.34	33097.42
3	23843.01	52034.82	24982.03	54409.88	52259.7	52066.18	51940.66	52021.88	52034.52	22748.27	49637.38
4	31790.68	69379.76	33308.69	72545.15	69679.86	69421.61	69254.42	69362.54	69379.36	30326.93	66179.08
5	39738.35	86724.7	41635.47	90680.65	87099.98	86777.03	86568.15	86703.19	86724.19	37906.30	82721.49

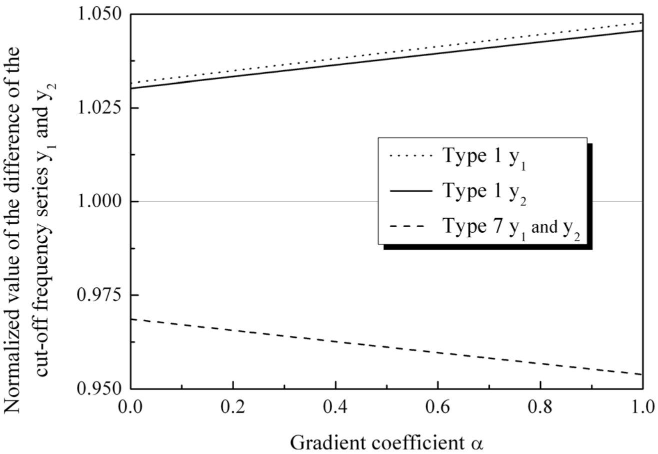

FIG. 2. Normalized value of the difference of the cut-off frequency series y_1 and y_2 plotted as a function of gradient coefficient.

analytical solutions for the cut-off frequencies of Lamb waves in a homogenous thin film.[2] Comparing the result for the numerical example published in Ref. 9 with the results obtained by Eqs. (13) and (14), we find they are in full agreement.

We provide an example to show the effect of the gradient properties on the cut-off frequencies of Lamb waves in an FGPPM plate. FGPPM parameters are assumed to be in the following form:

$$f(z) = f(0) + \alpha[f(h) - f(0)]\frac{z}{h} + (1-\alpha)[f(h) - f(0)]\left(\frac{z}{h}\right)^2,$$

where α is the graded coefficient; f_0 and f_h stand for the material parameters at $z = 0$ and $z = h$, respectively. In the following calculation, some values of parameters f_0 are chosen to be[6] $c_{33} = 218GPa$, $c_{44} = 48GPa$, $e_{33} = 6.9C/m^2$, $f_{33} = 358N/Am$, $\varepsilon_{33} = 5.1 \times 10^{-9}C/Vm$, $\mu_{33} = 98 \times 10^{-6}Ns^2/C^2$, $g_{33} = 2.73 \times 10^{-9}Ns/VC$, and $\rho = 7.5 \times 10^3 kg/m^3$. A relative variation coefficient of the material parameters, $\delta(f) = (f_h - f_0)/f_0$, and seven different types of material variations are discussed: type 1: $\delta(c_{33}) = \delta(c_{44}) = 0.2$; type 2: $\delta(e_{33}) = 0.2$; type 3: $\delta(f_{33}) = 0.2$; type 4: $\delta(\varepsilon_{33}) = 0.2$; type 5: $\delta(\mu_{33}) = 0.2$; type 6: $\delta(g_{33}) = 0.2$; and type 7: $\delta(\rho) = 0.2$. In these types, the unlisted material parameters are constants. When $\alpha = 1$, the material parameters vary along the thickness linearly. The first five values of $\omega_{nT}h$ and $\omega_{nL}h$ obtained by Eqs. (13) and (14), respectively, are discussed (Table I). The presence of a homogenous thin film indicates that all parameters are constants so that $f(z) = f(0)$.

In types 2–6, the material gradient property does not affect the series of cut-off frequencies $\omega_{nT}h$ so that only the series of $\omega_{nL}h$ in these structures are listed in Table I. The series of $\omega_{nT}h$ in the structures made of types 2–6 is the same as that in the homogenous piezoelectric–piezomagnetic thin films. When c_{33}, c_{44}, e_{33}, and f_{33} increase with thickness, the cut-off frequencies in the FGPPM thin film become larger than those in the homogenous thin film. By contrast, when ε_{33}, μ_{33}, g_{33}, and ρ increase with thickness, the cut-off frequencies in the FGPPM thin film become smaller than those in the homogenous thin film. Table I also shows that for the same variation of material parameters, the variations in c_{33}, c_{44}, and mass density ρ significantly affect the cut-off frequencies. The variations in e_{33}, f_{33}, and ε_{33} affect the cut-off frequencies relatively less than do the others, whereas the variations in μ_{33} and g_{33} almost do not affect the cut-off frequencies at all.

Consider the application in NDE. The gradient coefficient can be evaluated by measuring the cut-off frequencies of Lamb waves in an FGPPM thin film. The set of cut-off frequencies can be considered a union of two series of approximate arithmetic progression, in which the difference is $\pi\sqrt{c_{440}/\rho_0}/h$ and $\pi\sqrt{c_{E0}/\rho_0}/h$, respectively. We define the normalized value of the difference as

$$y_1 = \left[\pi \bigg/ \int_0^h \sqrt{\rho/c_{44}}dz\right] \bigg/ \left[\pi\sqrt{c_0/\rho_0}\bigg/h\right],$$

$$y_2 = \left[\pi \bigg/ \int_0^h \sqrt{\rho/c_E}dz\right] \bigg/ \left[\pi\sqrt{c_0/\rho_0}\bigg/h\right].$$

Figure 2 shows the relationship between the gradient coefficient and normalized value of the difference of cut-off frequency series y_1 and y_2. The relationship between the difference of the series and gradient coefficient becomes almost linear. The result provides a method with which to determine the gradient coefficient in inhomogeneous thin films using Lamb waves.

In summary, when the variation in material parameters is continuous and derivable, and the surface of the film is traction-free, unelectroded, and unmagnetized, regardless of how complicated the variation in material parameters along the thickness and multi-field couples in the material is, the cut-off frequencies of Lamb waves in these inhomogeneous plates can be solved using two simple equations which are related to $\sqrt{\rho/c_{44}}$ and $\sqrt{\rho/c_E}$, respectively. This study also reveals that the elastic parameters and density significantly affect cut-off frequencies, some piezoelectric and piezomagnetic parameters affect cut-off frequencies relatively less than do others, and the others do not or almost do not affect the cut-off frequencies.

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