Fractional order generalized electro-magneto-thermo-elasticity

Article *in* European Journal of Mechanics - A/Solids · November 2013

DOI: 10.1016/j.euromechsol.2013.05.006

3 authors:


Yajun Yu
Department of Solid Mechanics, SVL, Xi'an Ji…


Xiaogeng Tian
Xi'an Jiaotong University


Tian Jian Lu
Xi'an Jiaotong University

European Journal of Mechanics A/Solids 42 (2013) 188–202

Contents lists available at SciVerse ScienceDirect

European Journal of Mechanics A/Solids

journal homepage: www.elsevier.com/locate/ejmsol

Fractional order generalized electro-magneto-thermo-elasticity

Ya Jun Yu, Xiao Geng Tian*, Tian Jian Lu

State Key Laboratory for Mechanical Structure and Vibration, Xi'an Jiaotong University, Xi'an 710049, PR China

ARTICLE INFO

Article history:
Received 16 November 2012
Accepted 29 May 2013
Available online 10 June 2013

Keywords:
Generalized electro-magneto-thermo-elasticity
Variational theorem
Fractional calculus

ABSTRACT

Built upon the fractional order generalized thermoelasticity (FOGTE), which is based on ETE (extended thermoelasticity), a fractional order generalized electro-magneto-thermo-elasticity (FOGEMTE) theory is developed for anisotropic and linearly electro-magneto-thermo-elastic media by introducing the dynamic electro-magnetic fields, with various generalized thermoelasticity considered, such as ETE, TRDTE (temperature rate dependent thermoelasticity), TEWOED (thermoelasticity without energy dissipation), TEWED (thermoelasticity with energy dissipation), DPLTE (dual-phase-lag thermoelasticity). The two temperature (thermodynamics and conductive temperature) model is also introduced. In addition, to numerically deal with the multi-physics problems expressed by a series of partial differential equations especially a fractional one, the corresponding variational principle based on the variational integral method is proposed, and various degenerated variational theorems are presented. A generalized variational theorem is obtained for the unified theory by using the semi-inverse method. Finally, two examples are numerically validated, and concluding remarks are also given.

1. Introduction

The classical coupled theory of thermoelasticity (CTE) developed by Biot (1956) predicts an infinite speed of propagation for thermal signals, which however contradicts physical facts (Peshkov, 1944). To eliminate the paradox, a number of generalized theories involving a finite speed of heat conduction have been proposed. The first of such theory is the extended thermoelasticity theory (ETE) proposed by Lord and Shulman (1967) by introducing one thermal relaxation time into the classical Fourier law of heat conduction. Green and Lindsay (1972) proposed another theory by modifying the stress–strain relationship as well as the entropy relationship with relaxation time, which is known as the temperature rate dependent thermoelasticity (TRDTE). Subsequently, Green and Naghdi (1991, 1992, 1993) introduced the theory of thermoelasticity without energy dissipation (TEWOED) and the theory of thermoelasticity with energy dissipation (TEWED). There exist also other generalized thermoelasticity theories such as the two-temperature generalized thermoelasticity (TTGTE) (Youssef, 2006), the low-temperature thermoelasticity (Hetnarski and Ignaczak, 1993, 1994), the dual-phase-lag thermoelasticity (DPLTE) (Tzou, 1995), and the three-phase-lag thermoelasticity (TPLTE) (Roychoudhuri, 2007).

On a separate front, investigations concerning fractional derivatives and fractional integrals have become increasingly important as numerous models in the field of chemistry, physics, aerodynamics, etc. are now expressed in terms of fractional order. Whilst many theoretical studies concerning, e.g., the existence and uniqueness of solution of fractional differential equations, have been carried out (see, e.g., Bai and Lu, 2005; Deng and Ma, 2010; Su, 2009), numerical algorithms essential for dealing with engineering problems in the context of fractional derivatives and fractional integrals have also been proposed (Diethelm et al., 2005).

There exist many materials and physical situations where classical and above-mentioned generalized thermoelasticity, breaks down: low temperature regimes, amorphous media, colloids, glassy and porous materials, man-made and biological materials or polymers, transient loading, etc. In such cases, one may need to introduce time-fractional calculus into thermoelasticity (Ignaczak and Ostoja-Starzewski, 2010). Recently, upon introducing the Riemann–Liouville fractional integral operator into the generalized law of heat conduction, Youssef (2010a, 2010b) formulated the theory of fractional order generalized thermoelasticity (FOGTE) and established the corresponding variational principle. The theory is subsequently employed to solve two-dimensional thermal shock problems with Laplace and Fourier transforms (Youssef, 2012) as well as half-space problems of elastic materials subjected to ramp-type heating by using Laplace transform and state-space methods (Youssef and Al-Lehaibi, 2010). Abouelregal (2011) also established a model of fractional order generalized thermo-piezoelectricity and

* Corresponding author.
E-mail address: tiansu@mail.xjtu.edu.cn (X.G. Tian).

0997-7538/$ – see front matter
http://dx.doi.org/10.1016/j.euromechsol.2013.05.006

used it to solve one-dimensional boundary value problems for semi-infinite piezoelectric media. It is noted that another fractional generalized heat conduction law has also been proposed (Ezzat, 2011; Ezzat and Karamany, 2011a, 2011b, 2011c; Sherief et al., 2010).

Recent development of materials science has enabled processing of novel functional materials based on coupled properties of multi-fields, attracting accordingly interests for multi-fields coupling problems. For instance, Niraula and Noda (2010) derived constitutive equations of electro-magneto-thermo-elasticity; Chen et al. (2004) obtained general solutions for transversely isotropic electro-magneto-thermo-elasticity using the operator theory, which can be further simplified by using the generalized Almansi's theorem. However, due to the complexity of the governing equations involved, analytical solutions for multi-fields coupling problems are typically difficult to obtain, and hence various variational principles have been proposed for the purpose of numerical calculations (He, 2001, 2002; Wang et al., 2002, 2010).

Whilst a number of problems in generalized thermoelasticity have been solved by different means such as the state space method and integral transform technique, Tian et al. (2006) made a breakthrough by directly solving the governing equations in time domain using the finite element method (FEM). They successfully predicted the delicate second sound effect of heat conduction, which is difficult to model by other methods. FEM is an effective way to address these problems, therefore the variational theorem (or virtual work principle) as the foundation of FEM is of great importance. To establish the variational theorem, the method of variational integral is frequently employed (Liang et al., 2005) due to its wide applicability. In addition, the semi-inverse method proposed by He (1997) also proves to be convenient and effective.

It appears that the theory of fractional order generalized thermoelasticity coupled with electric and magnetic fields (in particular fully dynamic multi-fields coupling) and the corresponding variational theorem have not been established. To address this deficiency, in the present investigation, the theory of fractional order generalized electro-magneto-thermoelasticity (FOGEMTE) for anisotropic and linearly electro-magneto-thermo-elastic media is developed by considering dynamic electric and magnetic fields. In addition, Youssef's work (2010a, 2010b) is only based on the ETE theory, while in this work the concept of fractional calculus is extended into TRDTE, TEWOED, TEWED and DPL, of which a general form is introduced during the formulation. The corresponding variational theorem and generalized variational principle are also presented, and some numerical examples are attached to validate the proposed models.

2. General background

As an essential part of (generalized) thermoelasticity theory, the theory of (generalized) heat transfer plays a significant role in the investigation of thermoelasticity. In this section, we first summarize the heat conduction law, including Fourier's law and its modified versions called generalized or hyperbolic heat conduction law. Then, we formulated a unified form of heat conduction law, which will be used in Section 3 to develop FOGEMTE.

In 1807, Fourier formulated the transient process of heat conduction in the form of partial differential equation, whereas the unprecedented work was published until 1822. After publication, the power and significance of Fourier's work has been spread into many fields besides heat transfer, such as electricity, chemical diffusion, genetics, etc. In modern notation, the parabolic partial differential equation may be written as:

$$k\nabla^2\theta = \rho c_{\mathrm{E}}\dot{\theta} \tag{1}$$

which can be obtained by eliminating $\mathbf{q}$ between the Fourier's heat conduction law (or HCL_1 in current notation):

$$\mathbf{q} = -k\nabla\theta \tag{2}$$

and the equation for energy conservation

$$\nabla\cdot\mathbf{q} = -\rho T_0\dot{\eta} \tag{3}$$

where k, θ, ρ, c_{E}, $\mathbf{q}$, T_0, η are the coefficient of thermal conductivity, relative temperature, mass density, specific heat, heat flux vector, reference temperature and entropy density, respectively. ∇ is the gradient operator. θ is related to the temperature T, as:

$$\theta = T - T_0 \tag{4}$$

Obviously, Eq. (1) has an unphysical feature that is the infinite velocity of heat signals. To remove this, some nonclassical models have been developed. The first of such model is often referred as Cattaneo–Vernotte (CV) equation (HCL_2) (Cattaneo, 1958; Vernotte, 1958a, 1958b, 1961):

$$\mathbf{q} + \tau\dot{\mathbf{q}} = -k\nabla\theta \tag{5}$$

where τ is the relaxation time, specifically the time lag between the temperature gradient and the resulting heat flux vector. The corresponding heat conduction equation reads as:

$$k\nabla^2\theta = \rho c_{\mathrm{E}}\left(\dot{\theta} + \tau\ddot{\theta}\right) \tag{6}$$

which predicts a finite velocity as $v_\theta = \sqrt{k/\rho c_{\mathrm{E}}\tau}$.

As aforementioned in Section 1, Green and Naghdi (1991, 1992, 1993) formulated the theory of TEWOED and TEWED, but it is needed to illustrate that the procedure used by Green and Naghdi is significantly different from the conventional approach: the latter is upon the equation of balance of energy and an entropy production inequality, while the former is upon a reduced equation of balance of energy as:

$$\rho\dot{\eta} = \rho(s + \xi) - p_{k,k} \tag{7}$$

which was introduced in their early literature (Green and Naghdi, 1977). Later, some researchers (Mallik and Kanoria, 2008; Taheri et al., 2005) considered TEWOED and TEWED with the conventional approach and proposed another modified version of Fourier's law (here we refer to it as HCL_3):

$$\dot{\mathbf{q}} = -\left(k\nabla\dot{\theta} + k^*\nabla\theta\right) \tag{8}$$

which has turned out to be rational. In Eq. (8) k^* is a material constant characteristic. In his review, Chandrasekharaiah (1998) also discussed the issue in detail. Eq. (8) has the form if k is identically zero, as:

$$\dot{\mathbf{q}} = -k^*\nabla\theta \tag{9}$$

Joseph and Preziosi (1989) referred to the theory based on Eq. (9) as inertial theory of heat conduction (HCL_4). Actually, Eq. (9) is a special form of Eq. (5) in the case that τ tends to infinity while k/τ is finite.

Tzou (1995) proposed an alternative generalization of Fourier's law, as:

$$\mathbf{q}(t + \tau_q) = -k\nabla\theta(t + \tau_\theta) \tag{10}$$

where τ_q is phase lag of the heat flux caused by the fast transient effects of thermal inertia, while τ_θ is the phase lag of the temperature gradient due to microstructural interactions. Expanding both sides of Eq. (10) by Taylor's series, we obtain the following heat conduction law:

$$\mathbf{q}+\tau_q\dot{\mathbf{q}} = -k\left(\nabla\theta+\tau_\theta\nabla\dot{\theta}\right) \tag{11}$$

$$\mathbf{q}+\tau_q\dot{\mathbf{q}}+0.5\tau_q^2\ddot{\mathbf{q}} = -k\left(\nabla\theta+\tau_\theta\nabla\dot{\theta}\right) \tag{12}$$

Here and after, we refer to Eq. (11) as Dual-phase-lag model of heat conduction (HCL_5).

In summary, a unified form of heat conduction law may be expressed, as:

$$w_1\mathbf{q}+(w_2\tau_1+w_3)\dot{\mathbf{q}} = -w_4k\nabla\theta-w_5k^*\nabla\theta-w_6k\nabla\dot{\theta}-w_7k\tau_2\nabla\dot{\theta} \tag{13}$$

which can be degenerated into Fourier's law and its modifications, as shown in Table 1.

3. Governing equations and general theory

Thermoelasticity is a fusion of the theory of heat conduction and elasticity, and in Section 2 we have reviewed the heat conduction law and formulated a unified form of heat conduction law, which will be utilized in this part to develop the unified theory of FOGEMTE, with the dynamic electromagnetic theory and fractional calculus introduced.

The equations of motion as well as charge and current conservation may be written, as:

$$\nabla\cdot\boldsymbol{\sigma}+\mathbf{f} = \rho\ddot{\mathbf{u}} \tag{14}$$

$$\nabla\cdot\mathbf{D}-\rho_e = 0 \tag{15}$$

$$\nabla\cdot\mathbf{B} = 0 \tag{16}$$

where $\boldsymbol{\sigma}$, $\mathbf{D}$, $\mathbf{B}$ are the stress tensor, electric displacement vector, and magnetic induction vector, respectively; $\mathbf{f}$, ρ_e are the body force and electric charge density; ρ is the mass density; $\mathbf{u}$ is the elastic displacement vector. Introducing the magnetic vector potential $\mathbf{A}$, one obtains from Eq. (16) that:

$$\mathbf{B} = \nabla\times\mathbf{A} \tag{17}$$

The generalized forms of strain-displacement relations, electric intensity and magnetic intensity are given by:

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left(\nabla\mathbf{u}+\nabla\mathbf{u}^T\right) \tag{18}$$

$$\nabla\times\mathbf{E} = -\dot{\mathbf{B}} \tag{19}$$

$$\nabla\times\mathbf{H} = \dot{\mathbf{D}}+\mathbf{J} \tag{20}$$

Table 1
The heat conduction law and corresponding value of w_i.

	Eq. number	w_1	w_2	w_3	w_4	w_5	w_6	w_6	Supplementary conditions
HCL_1	2	1	0	0	1	0	0	0	
HCL_2	5	1	1	0	1	0	0	0	$\tau_1=\tau$
HCL_3	8	0	0	1	0	1	1	0	
HCL_4	9	0	0	1	0	1	0	0	
HCL_5	11	1	1	0	1	0	0	1	$\tau_1=\tau_q$, $\tau_2=\tau_\theta$

where $\boldsymbol{\varepsilon}$, $\mathbf{E}$, $\mathbf{H}$, $\mathbf{J}$ are the strain, electric intensity, magnetic intensity and current density, respectively.

Substituting (17) into (19) and introducing the electric potential φ, one arrives at:

$$\mathbf{E} = -\nabla\varphi-\dot{\mathbf{A}} \tag{21}$$

Analogy to Youssef (2010a), we introduce the Riemann–Liouville fractional integral operator into the unified form of heat conduction law Eq. (13), as:

$$w_1\mathbf{q}+(w_2\tau_1+w_3)\dot{\mathbf{q}} = -w_4kI^{\alpha-1}\nabla\theta-w_5k^*I^{\alpha-1}\nabla\theta - w_6kI^{\alpha-1}\nabla\dot{\theta}-w_7k\tau_2I^{\alpha-1}\nabla\dot{\theta} \tag{22}$$

where θ is called conductive temperature to distinguish from the thermodynamics temperature, and I indicates an integral operator defined as (Podlubny, 1999; Mainardi and Gorenflo, 2000):

$$I^{\alpha-1}f(t) = \frac{1}{\Gamma(\alpha-1)}\int_0^t(t-\tau)^{\alpha-2}f(\tau)\mathrm{d}\tau \tag{23}$$

where $\Gamma(\alpha)$ is the gamma function, $0<\alpha\leq 2$, and

$$I^0f(t) = f(t),\quad I^{-\alpha}f(t) = \frac{\partial^\alpha}{\partial t^\alpha}f(t) \tag{24}$$

In the absence of any inner heat source, the equation for energy conservation is:

$$\nabla\cdot\mathbf{q} = -\rho T_0\dot{\eta} \tag{25}$$

For anisotropic and linearly electro-magneto-thermo-elastic media, the coupled constitutive relations may be described as (Chen et al., 2006; Tian et al., 2007):

$$\boldsymbol{\sigma} = \mathbf{c}\boldsymbol{\varepsilon}-\mathbf{d}\mathbf{E}-\mathbf{b}\mathbf{H}-\boldsymbol{\chi}^{\boldsymbol{\sigma}}\left(\vartheta+w_8\tau_3\dot{\vartheta}\right) \tag{26}$$

$$\mathbf{D} = \mathbf{d}\boldsymbol{\varepsilon}+\boldsymbol{\alpha}\mathrm{E}+\boldsymbol{\zeta}\mathrm{H}+\boldsymbol{\chi}^{\mathbf{D}}\left(\vartheta+w_8\tau_3\dot{\vartheta}\right) \tag{27}$$

$$\mathbf{B} = \mathbf{b}\boldsymbol{\varepsilon}+\boldsymbol{\zeta}\mathrm{E}+\boldsymbol{\beta}\mathrm{H}+\boldsymbol{\chi}^{\mathbf{B}}\left(\vartheta+w_8\tau_3\dot{\vartheta}\right) \tag{28}$$

$$\rho\eta = \boldsymbol{\chi}^{\boldsymbol{\sigma}}:\boldsymbol{\varepsilon}+\boldsymbol{\chi}^{\mathbf{D}}\cdot\mathbf{E}+\boldsymbol{\chi}^{\mathbf{B}}\cdot\mathbf{H}+\frac{\rho c_E}{T_0}\left(\vartheta+w_8\tau_4\dot{\vartheta}\right) \tag{29}$$

where $\mathbf{c}$, $\boldsymbol{\alpha}$, and $\boldsymbol{\beta}$ are the elastic, dielectric, magnetic permeability coefficients, respectively; $\mathbf{d}$, $\mathbf{b}$, and $\boldsymbol{\zeta}$ are the piezoelectric, piezomagnetic and magneto-electric coefficients, respectively; $\boldsymbol{\chi}^{\boldsymbol{\sigma}}$, $\boldsymbol{\chi}^{\mathbf{D}}$, and $\boldsymbol{\chi}^{\mathbf{B}}$ are thermal modulus, pyroelectric and pyromagnetic constants, respectively; τ_3 and τ_4 are the relaxation times in TRDTE, and ϑ is the thermodynamic temperature related to the conductive temperature via:

$$\vartheta = \theta-w_9a\nabla^2\theta \tag{30}$$

here, a is the two-temperature parameter.

As fundamental equations governing the force field, the electric field, the magnetic field and the temperature field, Eqs. (14)–(30) are deterministic, totalling 36 equations for 36 unknown variables. Substituting Eq. (26) into (14), and then considering Eqs. (18) and (21), one obtains the governing equation of the force field, as:

$$\nabla\cdot\left\{0.5\mathbf{c}\left(\nabla\mathbf{u}+\nabla\mathbf{u}^{T}\right)-\mathbf{d}\left(-\nabla\varphi-\dot{\mathbf{A}}\right)-\mathbf{b}\mathbf{H}-\boldsymbol{\chi}^{\boldsymbol{\sigma}}\left[\left(\theta-w_9 a\theta_{,ii}\right)+w_8\tau_3\left(\dot{\theta}-w_9 a\dot{\theta}_{,ii}\right)\right]\right\}+\mathbf{f}=\rho\ddot{\mathbf{u}} \tag{31}$$

Similarly, introducing (27) into (15) (or (28) into (16)) and considering (18) and (21) leads to the governing equations of the electric and magnetic fields, as:

$$\nabla\cdot\left\{0.5\mathbf{d}\left(\nabla\mathbf{u}+\nabla\mathbf{u}^{T}\right)+\boldsymbol{\alpha}\left(-\nabla\varphi-\dot{\mathbf{A}}\right)+\boldsymbol{\zeta}\mathrm{H}+\boldsymbol{\chi}^{\mathbf{D}}\left[\left(\theta-w_9 a\theta_{,ii}\right)+w_8\tau_3\left(\dot{\theta}-w_9 a\dot{\theta}_{,ii}\right)\right]\right\}-\rho_e=0 \tag{32}$$

$$\nabla\cdot\left\{0.5\mathbf{b}\left(\nabla\mathbf{u}+\nabla\mathbf{u}^{T}\right)+\boldsymbol{\zeta}\left(-\nabla\varphi-\dot{\mathbf{A}}\right)+\boldsymbol{\beta}\mathrm{H}+\boldsymbol{\chi}^{\mathbf{B}}\left[\left(\theta-w_9 a\theta_{,ii}\right)+w_8\tau_3\left(\dot{\theta}-w_9 a\dot{\theta}_{,ii}\right)\right]\right\}=0 \tag{33}$$

The governing equation for the temperature field is obtained by first combining (22), (24) and (25) and then introducing (29), as:

$$T_0\beta\left\{0.5\boldsymbol{\chi}^{\boldsymbol{\sigma}}:\left(\nabla\mathbf{u}+\nabla\mathbf{u}^{T}\right)+\boldsymbol{\chi}^{\mathbf{D}}\cdot\left(-\nabla\varphi-\dot{\mathbf{A}}\right)+\boldsymbol{\chi}^{\mathbf{B}}\cdot\mathbf{H}+\left(\rho c_E/T_0\right)\times\left[\left(\theta-w_9 a\theta_{,ii}\right)+w_8\tau_4\left(\dot{\theta}-w_9 a\dot{\theta}_{,ii}\right)\right]\right\}=w_4 k\nabla^2\theta+w_5 k^{*}\nabla^2\theta+w_6 k\nabla^2\dot{\theta}+w_7 k\tau_2\nabla^2\dot{\theta} \tag{34}$$

where

$$\beta=w_1\frac{\partial^{\alpha}}{\partial t^{\alpha}}+\left(w_2\tau_1+w_3\right)\frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}}$$

It is difficult to eliminate **H** when deriving Eqs. (31)–(34), and there is only one equation, i.e., Eq. (33), relating to the magnetic vector potential **A** that has three components for the magnetic field. Hence, equations (17) and (20) should be introduced as complementary equations. By considering Eqs. (18), (21) and (27), Eq. (20) is rewritten as follows for coherence:

$$\nabla\times\mathbf{H}=0.5\mathbf{d}\left(\nabla\dot{\mathbf{u}}+\nabla\dot{\mathbf{u}}^{T}\right)+\boldsymbol{\alpha}\left(-\nabla\dot{\varphi}-\ddot{\mathbf{A}}\right)+\boldsymbol{\zeta}\dot{\mathbf{H}}+\boldsymbol{\chi}^{\mathbf{D}}\times\left[\left(\dot{\theta}-w_9 a\dot{\theta}_{,ii}\right)+w_8\tau_3\left(\ddot{\theta}-w_9 a\ddot{\theta}_{,ii}\right)\right]+\mathbf{J} \tag{35}$$

Note that Eq. (17) indicates three equations, of which any two are independent given Eq. (16).

In summary, there are eleven independent equations, i.e., Eqs. (17) and (31)–(35), corresponding to eleven unknown quantities, i.e., **u**, φ, **A**, **H** and θ, and all of these quantities can be determined upon introducing boundary conditions and initial conditions. On surfaces s_1 and s_2, the displacement and traction are given as:

$$\begin{aligned}\mathbf{u}&=\overline{\mathbf{u}}\quad\text{on surface } s_1\\ \boldsymbol{\sigma}\cdot\mathbf{n}&=\mathbf{F}\quad\text{on surface } s_2\end{aligned} \tag{36}$$

On surfaces s_3 and s_4, the electric potential and surface charges are expressed as:

$$\begin{aligned}\varphi&=\overline{\varphi}\quad\text{on surface } s_3\\ \mathbf{D}\cdot\mathbf{n}&=\overline{d}\quad\text{on surface } s_4\end{aligned} \tag{37}$$

On surfaces s_5 and s_6, the boundary conditions are prescribed as:

$$\begin{aligned}\mathbf{A}&=\overline{\mathbf{A}}\quad\text{on surface } s_5\\ \nabla\times\mathbf{H}&=\overline{\mathbf{h}}\quad\text{on surface } s_6\end{aligned} \tag{38}$$

On surface s_7 and s_8, the temperature and heat flux are given as:

$$\begin{aligned}\theta&=\overline{\theta}\quad\text{on surface } s_7\\ \mathbf{q}\cdot\mathbf{n}&=\overline{q}\quad\text{on surface } s_8\end{aligned}$$

Note that, in the above boundary conditions, $s_1+s_2=s_3+s_4=s_5+s_5=s_7+s_8=s$ covers the total boundary surface.

Thus far we have perfected the FOGTE theory, which is only based on ETE, and developed the FOGEMTE theory by introducing the dynamic electric and magnetic fields, with various generalized thermoelasticity considered (see, e.g. ETE, TRDTE, TEWOED, TEWED, DPLTE as shown in Table 2). In addition, the thermodynamics temperature and conductive temperature are distinguished if w_9 is set as 1.

4. Variational theorem

According to the formulation by Youssef (2010a), substituting Eq. (36) into the statement of virtual external work:

$$\int_v \mathbf{f}\cdot\delta\mathbf{u}\,\mathrm{d}v+\int_{s_\sigma}\mathbf{F}\cdot\delta\mathbf{u}\,\mathrm{d}s$$

and employing the divergence theorem, and then considering Eqs. (14) and (18), one obtains the following force field:

$$\int_v \boldsymbol{\sigma}:\delta\boldsymbol{\varepsilon}\,\mathrm{d}v+\int_v \rho\ddot{\mathbf{u}}\cdot\delta\mathbf{u}\,\mathrm{d}v=\int_v \mathbf{f}\cdot\delta\mathbf{u}\,\mathrm{d}v+\int_{s_\sigma}\mathbf{F}\cdot\delta\mathbf{u}\,\mathrm{d}s \tag{39}$$

This, together with Eq. (26), yields:

$$\int_v\left\{\mathbf{c}\boldsymbol{\varepsilon}-\mathbf{d}\mathbf{E}-\mathbf{b}\mathbf{H}-\boldsymbol{\chi}^{\boldsymbol{\sigma}}\left[\left(\theta-w_9 a\nabla^2\theta\right)+w_8\tau_3\left(\dot{\theta}-w_9 a\nabla^2\dot{\theta}\right)\right]\right\}:\delta\boldsymbol{\varepsilon}\,dv+\int_v\rho\ddot{\mathbf{u}}\cdot\delta\mathbf{u}\,\mathrm{d}v=\int_v\mathbf{f}\cdot\delta\mathbf{u}\,\mathrm{d}v+\int_{s_\sigma}\mathbf{F}\cdot\delta\mathbf{u}\,\mathrm{d}s \tag{40}$$

Table 2
Illustration and degeneration of FOGEMTE theory.

Theory	Based on		Conditions		
	Generalized thermoelasticity	Law of heat conduction	$w_i=1$	Fractional order	E-M field
FOGEMTE	ETE	HCL_2	$i=1,2,4$	Included	Included
	TRDTE		$i=1,4,8$		
	TEWOED	HCL_3	$i=3,5$		
	TEWED	HCL_4	$i=3,5,6$		
	DPLTE	HCL_5	$i=1,2,4,7$		
	If $w_9=1$, two-temperature (e.g. thermodynamics and conductive temperature) models are included.				
FOGTE	ETE	HCL_2	$i=1,2,4$	Included	Excluded
	TRDTE		$i=1,4,8$		
	TEWOED	HCL_3	$i=3,5$		
	TEWED	HCL_4	$i=3,5,6$		
	DPLTE	HCL_5	$i=1,2,4,7$		
ETE		HCL_2	$i=1,2,4$	Excluded	Excluded
TRDTE			$i=1,4,8$		
TEWOED		HCL_3	$i=3,5$		
TEWED		HCL_4	$i=3,5,6$		
DPLTE		HCL_5	$i=1,2,4,7$		

from which one obtains:

$$\begin{aligned}\int_v \boldsymbol{\chi}^{\boldsymbol{\sigma}}\theta : \delta\boldsymbol{\varepsilon} dv = & \int_v \mathbf{c}\boldsymbol{\varepsilon} : \delta\boldsymbol{\varepsilon} dv - \int_v \mathbf{dE} : \delta\boldsymbol{\varepsilon} dv - \int_v \mathbf{bH} : \delta\boldsymbol{\varepsilon} dv \\ & + \int_v Z_1 \boldsymbol{\chi}^{\boldsymbol{\sigma}}\nabla^2\theta : \delta\boldsymbol{\varepsilon} dv - \int_v Z_2 \boldsymbol{\chi}^{\boldsymbol{\sigma}}\dot{\theta} : \delta\boldsymbol{\varepsilon} dv \\ & + \int_v Z_3 \boldsymbol{\chi}^{\boldsymbol{\sigma}}\nabla^2\dot{\theta} : \delta\boldsymbol{\varepsilon} dv + \int_v \rho\ddot{\mathbf{u}} \cdot \delta\mathbf{u} dv \\ & - \int_v \mathbf{f}\cdot\delta\mathbf{u} dv - \int_{s_\sigma} \mathbf{F}\cdot\delta\mathbf{u} ds \end{aligned} \tag{41}$$

where

$Z_1 = w_9 a, \quad Z_2 = w_8\tau_3, \quad Z_3 = w_8 w_9 \tau_3 a$

In view of Eqs. (15) and (37), using the variational integral method, one arrives at:

$$-\int_v (\nabla\cdot\mathbf{D} - \rho_e)\delta\varphi dv + \int_{s_\mathbf{D}} \left(\mathbf{D}\cdot\mathbf{n} - \overline{d}\right)\delta\varphi ds = 0 \tag{42}$$

Integrating Eq. (42) by parts leads to:

$$\int_v \mathbf{D}\cdot\delta\nabla\varphi dv + \int_v \rho_e\delta\varphi dv = \int_{s_\mathbf{D}} \overline{d}\delta\varphi ds \tag{43}$$

Substitution of Eq. (19) into Eq. (43) yields:

$$-\int_v \mathbf{D}\cdot\delta\mathbf{E} dv - \int_v \mathbf{D}\cdot\delta\dot{\mathbf{A}} dv + \int_v \rho_e\delta\varphi dv = \int_{s_\mathbf{D}} \overline{d}\delta\varphi ds \tag{44}$$

Substitution of Eq. (27) into Eq. (44) yields:

$$\begin{aligned}& -\int_v \left\{\mathbf{d}\boldsymbol{\varepsilon} + \boldsymbol{\alpha}\mathrm{E} + \boldsymbol{\zeta}\mathrm{H} + \boldsymbol{\chi}^{\mathbf{D}}\left[\left(\theta - w_9 a\nabla^2\theta\right) + w_8\tau_3\left(\dot{\theta} - w_9 a\nabla^2\dot{\theta}\right)\right]\right\}\cdot\delta\mathbf{E} dv \\ & -\int_v \left\{\mathbf{d}\boldsymbol{\varepsilon} + \boldsymbol{\alpha}\mathrm{E} + \boldsymbol{\zeta}\mathrm{H} + \boldsymbol{\chi}^{\mathbf{D}}\left[\left(\theta - w_9 a\nabla^2\theta\right) + w_8\tau_3\left(\dot{\theta} - w_9 a\nabla^2\dot{\theta}\right)\right]\right\}\cdot\delta\dot{\mathbf{A}} dv \\ & + \int_v \rho_e\delta\varphi dv = \int_{s_\mathbf{D}} \overline{d}\delta\varphi ds \end{aligned} \tag{45}$$

which can be rewritten as:

$$\begin{aligned}\int_v \boldsymbol{\chi}^{\mathbf{D}}\theta\cdot\delta\mathbf{E} dv = & -\int_v \mathbf{d}\boldsymbol{\varepsilon}\cdot\delta\mathbf{E} dv - \int_v \boldsymbol{\alpha}\mathrm{E}\cdot\delta\mathbf{E} dv - \int_v \boldsymbol{\zeta}\mathrm{H}\cdot\delta\mathbf{E} dv \\ & + \int_v Z_1\boldsymbol{\chi}^D\nabla^2\theta\cdot\delta\mathbf{E} dv - \int_v Z_2\boldsymbol{\chi}^{\mathbf{D}}\dot{\theta}\cdot\delta\mathbf{E} dv \\ & + \int_v Z_3\boldsymbol{\chi}^{\mathbf{D}}\nabla^2\dot{\theta}\cdot\delta\mathbf{E} dv - \int_v \mathbf{d}\boldsymbol{\varepsilon}\cdot\delta\dot{\mathbf{A}} dv - \int_v \boldsymbol{\alpha}\mathrm{E}\cdot\delta\dot{\mathbf{A}} dv \\ & - \int_v \boldsymbol{\zeta}\mathrm{H}\cdot\delta\dot{\mathbf{A}} dv - \int_v \boldsymbol{\chi}^{\mathbf{D}}\theta\cdot\delta\dot{\mathbf{A}} dv + \int_v Z_1\boldsymbol{\chi}^{\mathbf{D}}\nabla^2\theta\cdot\delta\dot{\mathbf{A}} dv \\ & - \int_v Z_2\boldsymbol{\chi}^{\mathbf{D}}\dot{\theta}\cdot\delta\dot{\mathbf{A}} dv + \int_v Z_3\boldsymbol{\chi}^{\mathbf{D}}\nabla^2\dot{\theta}\cdot\delta\dot{\mathbf{A}} dv \\ & + \int_v \rho_e\delta\varphi dv - \int_{s_\mathbf{D}} \overline{d}\delta\varphi ds \end{aligned} \tag{46}$$

Using the variational integral method one obtains from Eqs. (17) and (38) that:

$$-\int_v (\mathbf{B} - \nabla\times\mathbf{A})\cdot\delta\mathbf{H} dv + \int_{s_\mathbf{A}} \left(\mathbf{A} - \overline{\mathbf{A}}\right)\cdot\delta(\nabla\times\mathbf{H}) ds = 0 \tag{47}$$

Integrating (35) by parts results in:

$$-\int_v \mathbf{B}\cdot\delta\mathbf{H} dv + \int_v \mathbf{A}\cdot\delta(\nabla\times\mathbf{H}) dv = \int_{s_\mathbf{A}} \overline{\mathbf{A}}\cdot\delta(\nabla\times\mathbf{H}) ds \tag{48}$$

Substitution of (20) into (48) leads to:

$$-\int_v \mathbf{B}\cdot\delta\mathbf{H} dv + \int_v \mathbf{A}\cdot\delta\dot{\mathbf{D}} dv + \int_v \mathbf{A}\cdot\delta\mathbf{J} dv = \int_{s_\mathbf{A}} \overline{\mathbf{A}}\cdot\delta(\nabla\times\mathbf{H}) ds \tag{49}$$

Substitution of (27) and (28) into (49) leads to:

$$\begin{aligned}& -\int_v \left\{\mathbf{b}\boldsymbol{\varepsilon} + \boldsymbol{\zeta}\mathrm{E} + \boldsymbol{\beta}\mathrm{H} + \boldsymbol{\chi}^{\mathbf{B}}\left[\left(\theta - w_9 a\nabla^2\theta\right) + w_8\tau_3\left(\dot{\theta} - w_9 a\nabla^2\dot{\theta}\right)\right]\right\}\cdot\delta\mathbf{H} dv \\ & + \int_v \mathbf{Ad} : \delta\boldsymbol{\varepsilon} dv + \int_v \mathbf{A}\boldsymbol{\alpha}\cdot\delta\dot{\mathbf{E}} dv + \int_v \mathbf{A}\boldsymbol{\zeta}\cdot\delta\dot{\mathbf{H}} dv + \int_v \mathbf{A}\cdot\boldsymbol{\chi}^{\mathbf{D}}\delta\dot{\theta} dv - w_9\int_v \mathbf{A}\cdot\boldsymbol{\chi}^{\mathbf{D}}a\delta\nabla^2\dot{\theta} dv \\ & + w_8\int_v \mathbf{A}\cdot\boldsymbol{\chi}^{\mathbf{D}}\tau_3\delta\ddot{\theta} dv - w_8 w_9\int_v \tau_3\mathbf{A}\cdot\boldsymbol{\chi}^{\mathbf{D}}a\delta\nabla^2\ddot{\theta} dv + \int_v \mathbf{A}\cdot\delta\mathbf{J} dv = \int_{s_\mathbf{A}} \overline{\mathbf{A}}\cdot\delta(\nabla\times\mathbf{H}) ds \end{aligned} \tag{50}$$

which can be rearranged as:

$$\begin{aligned}\int_v \boldsymbol{\chi}^{\mathbf{B}}\theta\cdot\delta\mathbf{H} dv = & -\int_v \mathbf{b}\boldsymbol{\varepsilon}\cdot\delta\mathbf{H} dv - \int_v \boldsymbol{\zeta}\mathrm{E}\cdot\delta\mathbf{H} dv - \int_v \boldsymbol{\beta}\mathrm{H}\cdot\delta\mathbf{H} dv + \int_v Z_1\boldsymbol{\chi}^{\mathbf{B}}\nabla^2\theta\cdot\delta\mathbf{H} dv - \int_v Z_2\boldsymbol{\chi}^{\mathbf{B}}\dot{\theta}\cdot\delta\mathbf{H} dv + \int_v Z_3\boldsymbol{\chi}^{\mathbf{B}}\nabla^2\dot{\theta}\cdot\delta\mathbf{H} dv \\ & + \int_v \mathbf{Ad} : \delta\boldsymbol{\varepsilon} dv + \int_v \mathbf{A}\boldsymbol{\alpha}\cdot\delta\dot{\mathbf{E}} dv + \int_v \mathbf{A}\boldsymbol{\zeta}\cdot\delta\dot{\mathbf{H}} dv + \int_v \mathbf{A}\cdot\boldsymbol{\chi}^{\mathbf{D}}\delta\dot{\theta} dv - \int_v Z_1\mathbf{A}\cdot\boldsymbol{\chi}^{\mathbf{D}}\delta\nabla^2\dot{\theta} dv + \int_v Z_2\mathbf{A}\cdot\boldsymbol{\chi}^{\mathbf{D}}\delta\ddot{\theta} dv \\ & - \int_v Z_3\mathbf{A}\cdot\boldsymbol{\chi}^{\mathbf{D}}\delta\nabla^2\ddot{\theta} dv + \int_v \mathbf{A}\cdot\delta\mathbf{J} dv - \int_{s_\mathbf{A}} \overline{\mathbf{A}}\cdot\delta(\nabla\times\mathbf{H}) ds \end{aligned} \tag{51}$$

Following Youssef (2010a), the entropy flux $\mathbf{S}$ is introduced as:

$$\mathbf{q} = T_0\dot{\mathbf{S}} \tag{52}$$

In view of Eq. (52), Eq. (25) can be rewritten as:

$$\rho\eta = -\nabla\mathbf{S} \tag{53}$$

Eliminating $\mathbf{q}$ between Eqs. (22) and (52), one gets:

$$T_0\beta\mathbf{S} + w_4k\nabla\theta + w_5k^*\nabla\theta + w_6k\nabla\dot{\theta} + w_7k\tau_2\nabla\dot{\theta} = 0 \tag{54}$$

where

$$\beta = w_1\frac{\partial^\alpha}{\partial t^\alpha} + (w_2\tau_1 + w_3)\frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}}$$

By multiplying δS_i and integrating over the body, it is obtained from Eq. (54) that:

$$\int_v \left(T_0\beta\mathbf{S} + w_4k\nabla\theta + w_5k^*\nabla\theta + w_6k\nabla\dot{\theta} + w_7k\tau_2\nabla\dot{\theta}\right)\cdot\delta\mathbf{S}\mathrm{d}v = 0 \tag{55}$$

Integrating (55) by parts leads to:

$$\int_v T_0\beta\mathbf{S}\cdot\delta\mathbf{S}\mathrm{d}v + \int_v Z_4\nabla(\theta\delta\mathbf{S})\mathrm{d}v - \int_v Z_4\theta\delta\nabla\mathbf{S}\mathrm{d}v + \int_v Z_5\nabla\dot{\theta}\cdot\delta\mathbf{S}\mathrm{d}v = 0 \tag{56}$$

where

$$Z_4 = w_4k + w_5k^*, \quad Z_5 = k(w_6 + w_7\tau_2)$$

Eliminating η in Eqs. (29) and (53) yields:

$$\nabla S = -\chi^{\sigma} : \varepsilon - \chi^{\mathbf{D}}\cdot\mathbf{E} - \chi^{\mathbf{B}}\cdot\mathbf{H} - (\rho c_{\mathrm{E}}/T_0)\Big[\left(\theta - w_9a\nabla^2\theta\right) + w_8\tau_4\left(\dot{\theta} - w_9a\nabla^2\dot{\theta}\right)\Big] \tag{57}$$

Substitution of Eq. (57) into (56) yields:

$$\begin{aligned}&\int_v T_0\beta\mathbf{S}\cdot\delta\mathbf{S}\mathrm{d}v + \int_v Z_4\nabla(\theta\delta\mathbf{S})\mathrm{d}v + \int_v Z_4\theta\chi^{\sigma} : \delta\varepsilon\mathrm{d}v + \int_v Z_4\theta\chi^{\mathbf{D}}\cdot\delta\mathbf{E}\mathrm{d}v + \int_v Z_4\theta\chi^{\mathbf{B}}\cdot\delta\mathbf{H}\mathrm{d}v + \int_v Z_4(\rho c_{\mathrm{E}}/T_0)\theta\delta\theta\mathrm{d}v \\ &- \int_v Z_1Z_4(\rho c_{\mathrm{E}}/T_0)\theta\delta\nabla^2\theta\mathrm{d}v + \int_v Z_4Z_6(\rho c_{\mathrm{E}}/T_0)\theta\delta\dot{\theta}\mathrm{d}v - \int_v Z_4Z_7(\rho c_{\mathrm{E}}/T_0)\theta a\delta\nabla^2\dot{\theta}\mathrm{d}v + \int_v Z_5\nabla\dot{\theta}\cdot\delta\mathbf{S}\mathrm{d}v = 0\end{aligned} \tag{58}$$

where

$$Z_6 = w_8\tau_4, \quad Z_7 = w_8w_9\tau_4a$$

As the entropy flux $\mathbf{S}$ is introduced to derive the temperature field, for clarity, the problem needs to be restated, as follows. With $\mathbf{S}$ included, there are fourteen unknown quantities, and Eq. (57) is taken as the additional complementary equations of governing Equations (17) and (31)–(35). By considering Eqs. (18) and (21), Eq. (57) is rewritten as:

$$\begin{aligned}\nabla\mathbf{S} = &-0.5\chi^{\sigma} : \left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right) - \chi^{\mathbf{D}}\cdot\left(-\nabla\varphi - \dot{\mathbf{A}}\right) - \chi^{\mathbf{B}}\cdot\mathbf{H} - (\rho c_{\mathrm{E}}/T_0)\theta \\ &+ Z_1(\rho c_{\mathrm{E}}/T_0)\nabla^2\theta + Z_2(\rho c_{\mathrm{E}}/T_0)\dot{\theta} + Z_3(\rho c_{\mathrm{E}}/T_0)\nabla^2\dot{\theta}\end{aligned} \tag{59}$$

In total, Eqs. (31)–(35) and (59) represent fourteen governing equations to be solved by considering boundary and initial conditions.

Finally, introducing Eqs. (41), (46) and (51) into Eq. (58), one obtains:

$$\begin{aligned}&\int_v Z_4\mathbf{c}\varepsilon : \delta\varepsilon\mathrm{d}v - \int_v Z_4\mathbf{dE} : \delta\varepsilon\mathrm{d}v - \int_v Z_4\mathbf{bH} : \delta\varepsilon\mathrm{d}v + \int_v Z_1Z_4\chi^{\sigma}\nabla^2\theta : \delta\varepsilon\mathrm{d}v - \int_v Z_2Z_4\chi^{\sigma}\dot{\theta} : \delta\varepsilon\mathrm{d}v + \int_v Z_3Z_4\chi^{\sigma}\nabla^2\dot{\theta} : \delta\varepsilon\mathrm{d}v \\
&- \int_v Z_4\mathbf{d}\varepsilon\cdot\delta\mathbf{E}\mathrm{d}v - \int_v Z_4\alpha\mathrm{E}\cdot\delta\mathbf{E}\mathrm{d}v - \int_v Z_4\zeta\mathrm{H}\cdot\delta\mathbf{E}\mathrm{d}v + \int_v Z_1Z_4\chi^{\mathbf{D}}\nabla^2\theta\cdot\delta\mathbf{E}\mathrm{d}v - \int_v Z_2Z_4\chi^{\mathbf{D}}\dot{\theta}\cdot\delta\mathbf{E}\mathrm{d}v + \int_v Z_3Z_4\chi^{\mathbf{D}}\nabla^2\dot{\theta}\cdot\delta\mathbf{E}\mathrm{d}v \\
&- \int_v Z_4\mathbf{d}\varepsilon\cdot\delta\dot{\mathbf{A}}\mathrm{d}v - \int_v Z_4\alpha\mathrm{E}\cdot\delta\dot{\mathbf{A}}\mathrm{d}v - \int_v Z_4\zeta\mathrm{H}\cdot\delta\dot{\mathbf{A}}\mathrm{d}v - \int_v Z_4\chi^{\mathbf{D}}\theta\cdot\delta\dot{\mathbf{A}}\mathrm{d}v + \int_v Z_1Z_4\chi^{\mathbf{D}}\nabla^2\theta\cdot\delta\dot{\mathbf{A}}\mathrm{d}v - \int_v Z_2Z_4\chi^{\mathbf{D}}\dot{\theta}\cdot\delta\dot{\mathbf{A}}\mathrm{d}v \\
&+ \int_v Z_3Z_4\chi^{\mathbf{D}}\nabla^2\dot{\theta}\cdot\delta\dot{\mathbf{A}}\mathrm{d}v - \int_v Z_4\mathbf{b}\varepsilon\cdot\delta\mathbf{H}\mathrm{d}v - \int_v Z_4\zeta\mathrm{E}\cdot\delta\mathbf{H}\mathrm{d}v - \int_v Z_4\beta\mathrm{H}\cdot\delta\mathbf{H}\mathrm{d}v + \int_v Z_1Z_4\chi^{\mathbf{B}}\nabla^2\theta\cdot\delta\mathbf{H}\mathrm{d}v - \int_v Z_2Z_4\chi^{\mathbf{B}}\dot{\theta}\cdot\delta\mathbf{H}\mathrm{d}v \\
&+ \int_v Z_3Z_4\chi^{\mathbf{B}}\nabla^2\dot{\theta}\cdot\delta\mathbf{H}\mathrm{d}v + \int_v Z_4\mathbf{Ad} : \delta\varepsilon\mathrm{d}v + \int_v Z_4\mathbf{A}\alpha\cdot\delta\dot{\mathbf{E}}\mathrm{d}v + \int_v Z_4\mathbf{A}\zeta\cdot\delta\dot{\mathbf{H}}\mathrm{d}v + \int_v Z_4\mathbf{A}\cdot\chi^{\mathbf{D}}\delta\dot{\theta}\mathrm{d}v - \int_v Z_1Z_4\mathbf{A}\cdot\chi^{\mathbf{D}}\delta\nabla^2\dot{\theta}\mathrm{d}v \\
&+ \int_v Z_2Z_4\mathbf{A}\cdot\chi^{\mathbf{D}}\delta\ddot{\theta}\mathrm{d}v - \int_v Z_3Z_4\mathbf{A}\cdot\chi^{\mathbf{D}}\delta\nabla^2\ddot{\theta}\mathrm{d}v + \int_v Z_4\mathbf{A}\cdot\delta\mathbf{J}\mathrm{d}v + \int_v T_0\beta\mathbf{S}\cdot\delta\mathbf{S}\mathrm{d}v + \int_v Z_4(\rho c_{\mathrm{E}}/T_0)\theta\delta\theta\mathrm{d}v \\
&- \int_v Z_1Z_4(\rho c_{\mathrm{E}}/T_0)\theta\delta\nabla^2\theta\mathrm{d}v + \int_v Z_4Z_6(\rho c_{\mathrm{E}}/T_0)\theta\delta\dot{\theta}\mathrm{d}v - \int_v Z_4Z_7(\rho c_{\mathrm{E}}/T_0)\theta\delta\nabla^2\dot{\theta}\mathrm{d}v + \int_v Z_5\nabla\dot{\theta}\cdot\delta\mathbf{S}\mathrm{d}v = -\int_v Z_4\rho\ddot{\mathbf{u}}\cdot\delta\mathbf{u}\mathrm{d}v \\
&+ \int_v Z_4\mathbf{f}\cdot\delta\mathbf{u}\mathrm{d}v + \int_{S_\sigma} Z_4\mathbf{F}\cdot\delta\mathbf{u}\mathrm{d}s - \int_v Z_4\rho_{\mathrm{e}}\delta\varphi\mathrm{d}v + \int_{S_D} Z_4\overline{d}\delta\varphi\mathrm{d}s + \int_{S_{\mathbf{A}}} Z_4\overline{\mathbf{A}}\cdot\delta(\nabla\times\mathbf{H})\mathrm{d}s - \int_v Z_4\nabla(\theta\delta\mathbf{S})\mathrm{d}v\end{aligned} \tag{60}$$

which represents the variational theorem of the FOGEMTE theory. Several degenerated forms can be obtained by neglecting the corresponding terms. For instance, by neglecting the magnetic field in Eq. (60), the variational principle of the fractional order generalized electro-thermoelectricity can be expressed as:

$$\begin{aligned}
&\int_v Z_4\mathbf{c}\boldsymbol{\varepsilon}:\delta\boldsymbol{\varepsilon}\mathrm{d}v-\int_v Z_4\mathbf{dE}:\delta\boldsymbol{\varepsilon}\mathrm{d}v+\int_v Z_1Z_4\boldsymbol{\chi}^{\sigma}\nabla^2\theta:\delta\boldsymbol{\varepsilon}\mathrm{d}v-\int_v Z_2Z_4\boldsymbol{\chi}^{\sigma}\dot{\theta}:\delta\boldsymbol{\varepsilon}\mathrm{d}v+\int_v Z_3Z_4\boldsymbol{\chi}^{\sigma}\nabla^2\dot{\theta}:\delta\boldsymbol{\varepsilon}\mathrm{d}v-\int_v Z_4\mathbf{d}\boldsymbol{\varepsilon}\cdot\delta\mathbf{E}\mathrm{d}v\\
&-\int_v Z_4\boldsymbol{\alpha}\mathrm{E}\cdot\delta\mathbf{E}\mathrm{d}v+\int_v Z_1Z_4\boldsymbol{\chi}^{\mathbf{D}}\nabla^2\theta\cdot\delta\mathbf{E}\mathrm{d}v-\int_v Z_2Z_4\boldsymbol{\chi}^{\mathbf{D}}\dot{\theta}\cdot\delta\mathbf{E}\mathrm{d}v+\int_v Z_3Z_4\boldsymbol{\chi}^{\mathbf{D}}\nabla^2\dot{\theta}\cdot\delta\mathbf{E}\mathrm{d}v+\int_v T_0\beta\mathbf{S}\cdot\delta\mathbf{S}\mathrm{d}v\\
&+\int_v Z_4(\rho c_{\mathrm{E}}/T_0)\theta\delta\theta\mathrm{d}v-\int_v Z_1Z_4(\rho c_{\mathrm{E}}/T_0)\theta\delta\nabla^2\theta\mathrm{d}v+\int_v Z_4Z_6(\rho c_{\mathrm{E}}/T_0)\theta\delta\dot{\theta}\mathrm{d}v-\int_v Z_4Z_7(\rho c_{\mathrm{E}}/T_0)\theta\delta\nabla^2\dot{\theta}\mathrm{d}v\\
&+\int_v Z_5\nabla\dot{\theta}\cdot\delta\mathbf{S}\mathrm{d}v=-\int_v Z_4\rho\ddot{\mathbf{u}}\cdot\delta\mathbf{u}\mathrm{d}v+\int_v Z_4\mathbf{f}\cdot\delta\mathbf{u}\mathrm{d}v+\int_{S_\sigma} Z_4\mathbf{F}\cdot\delta\mathbf{u}\mathrm{d}s-\int_v Z_4\rho_e\delta\varphi\mathrm{d}v+\int_{S_D} Z_4\overline{\mathrm{d}}\delta\varphi\mathrm{d}s-\int_v Z_4\nabla(\theta\delta\mathbf{S})\mathrm{d}v
\end{aligned}\tag{61}$$

Similarly, the variational principle of the fractional order generalized magneto-thermo-elasticity can be obtained by removing the electric field in Eq. (60), whilst the variational theorem of generalized thermoelasticity can be obtained by neglecting both the electric field and the magnetic field.

Upon neglecting the electric field, the magnetic field and the Riemann–Liouville fractional integral operator when $w_1=w_2=w_4=1$, Eq. (60) can be further degenerated into the variational theorem of FOGTE formulated by Youssef (2010a) for isotropic elastic medium, as:

$$\delta U+\delta P+\delta R=\delta W\tag{62}$$

where U, P, R and δW are separately the strain energy of the elastic medium, the heat potential (Naotak et al., 2003), the dissipation function (Naotak et al., 2003), and the generalized virtual external work, given by:

$$U=\frac{1}{2}\int_v \mathbf{c}\boldsymbol{\varepsilon}:\boldsymbol{\varepsilon}\mathrm{d}V\tag{63}$$

$$P=\frac{c_{\mathrm{E}}}{2T_0}\int_v \theta^2\mathrm{d}v\tag{64}$$

$$R=\frac{T_0\beta}{2k}\int_v \mathbf{S}\cdot\mathbf{S}\mathrm{d}v\tag{65}$$

$$\delta W=-\int_v \rho\ddot{\mathbf{u}}\cdot\delta\mathbf{u}\mathrm{d}v+\int_v \mathbf{f}\cdot\delta\mathbf{u}\mathrm{d}v+\int_{S_\sigma} \mathbf{F}\cdot\delta\mathbf{u}\mathrm{d}s-\int_v \nabla(\theta\delta\mathbf{S})\mathrm{d}v\tag{66}$$

By removing the temperature field in Eq. (60), the variational principle of electro-magneto-elasticity has the form:

$$\begin{aligned}
&\int_v Z_4\mathbf{c}\boldsymbol{\varepsilon}:\delta\boldsymbol{\varepsilon}\mathrm{d}v-\int_v Z_4\mathbf{dE}:\delta\boldsymbol{\varepsilon}\mathrm{d}v-\int_v Z_4\mathbf{bH}:\delta\boldsymbol{\varepsilon}\mathrm{d}v-\int_v Z_4\mathbf{d}\boldsymbol{\varepsilon}\cdot\delta\mathbf{E}\mathrm{d}v-\int_v Z_4\boldsymbol{\alpha}\mathrm{E}\cdot\delta\mathbf{E}\mathrm{d}v-\int_v Z_4\boldsymbol{\zeta}\mathrm{H}\cdot\delta\mathbf{E}\mathrm{d}v-\int_v Z_4\mathbf{d}\boldsymbol{\varepsilon}\cdot\delta\dot{\mathbf{A}}\mathrm{d}v-\int_v Z_4\boldsymbol{\alpha}\mathrm{E}\cdot\delta\dot{\mathbf{A}}\mathrm{d}v\\
&-\int_v Z_4\boldsymbol{\zeta}\mathrm{H}\cdot\delta\dot{\mathbf{A}}\mathrm{d}v-\int_v Z_4\mathbf{b}\boldsymbol{\varepsilon}\cdot\delta\mathbf{H}\mathrm{d}v-\int_v Z_4\boldsymbol{\zeta}\mathrm{E}\cdot\delta\mathbf{H}\mathrm{d}v-\int_v Z_4\boldsymbol{\beta}\mathrm{H}\cdot\delta\mathbf{H}\mathrm{d}v+\int_v Z_4\mathbf{Ad}:\delta\boldsymbol{\varepsilon}\mathrm{d}v+\int_v Z_4\mathbf{A}\boldsymbol{\alpha}\cdot\delta\dot{\mathbf{E}}\mathrm{d}v+\int_v Z_4\mathbf{A}\boldsymbol{\zeta}\cdot\delta\dot{\mathbf{H}}\mathrm{d}v\\
&+\int_v Z_4\mathbf{A}\cdot\delta\mathbf{J}\mathrm{d}v=-\int_v Z_4\rho\ddot{\mathbf{u}}\cdot\delta\mathbf{u}\mathrm{d}v+\int_v Z_4\mathbf{f}\cdot\delta\mathbf{u}\mathrm{d}v+\int_{S_\sigma} Z_4\mathbf{F}\cdot\delta\mathbf{u}\mathrm{d}s-\int_v Z_4\rho_e\delta\varphi\mathrm{d}v+\int_{S_D} Z_4\overline{\mathrm{d}}\delta\varphi\mathrm{d}s+\int_{S_{\mathbf{A}}} Z_4\overline{\mathbf{A}}\cdot\delta(\nabla\times\mathbf{H})\mathrm{d}s
\end{aligned}\tag{67}$$

Whilst several special and useful variational principles have been deduced based on Eq. (60), more specialized variational theorems can be obtained by choosing different values for $w_i (i=1,2,\ldots,9)$ as shown in Table 1.

5. Generalized variational theorem

The variational theorem for the newly proposed FOGEMTE is presented in Section 3. To provide a complete rationale for formulating numerical methods of the new FOGEMTE theory, we need to propose a generalized variational principle. In this section, we aim to formulate such a principle by using the semi-inverse method (He, 1997).

In 1964, Gurtin (1964) established dual-convolutional variational principles for elastic dynamics. Comparing with Hamilton-type variational theorems, Gurtin-type ones may provide all the characteristics of the initial boundary problems. But the discretization of Gurtin-type variational principle in time domain is complex, which greatly limits its applications to various dynamic problems. To avoid the Gurtin-type variational theorem, the time-derivative term in the basic equations should be expressed as (He, 2002):

$$\frac{\partial\psi}{\partial t} = \frac{\psi(t)-\psi(t_{n-1})}{\Delta t} \underset{\text{written as}}{=} \frac{\psi-\psi^{(n-1)}}{\Delta t} \tag{68}$$

where ψ indicates an arbitrary function, and $\Delta t = t - t_{n-1}$ is the equal step length. In view of (68), Eqs. (20), (21), and (25)–(28) can be rewritten sequentially as:

$$\nabla\times\mathbf{H} = Z_8\mathbf{D} - \mathbf{Z_9} + \mathbf{J} \tag{69}$$

$$\mathbf{E} = -\nabla\varphi - Z_8\mathbf{A} + \mathbf{Z}_{10} \tag{70}$$

$$\nabla\cdot\mathbf{q}\Delta t = -T_0\left(\boldsymbol{\chi}^{\boldsymbol{\sigma}}\boldsymbol{\varepsilon} + \boldsymbol{\chi}^{\mathbf{D}}\mathbf{E} + \boldsymbol{\chi}^{\mathbf{B}}\mathbf{H}\right) - \rho c_{\mathrm{E}}Z_{11}\theta + \rho c_{\mathrm{E}}Z_{12}\nabla^2\theta + Z_{13} \tag{71}$$

$$\boldsymbol{\sigma} = \mathbf{c}\boldsymbol{\varepsilon} - \mathbf{dE} - \mathbf{bH} - Z_{14}\boldsymbol{\chi}^{\boldsymbol{\sigma}}\theta + Z_{15}\boldsymbol{\chi}^{\boldsymbol{\sigma}}\nabla^2\theta + \mathbf{Z_{16}} \tag{72}$$

$$\mathbf{D} = \mathbf{d}\boldsymbol{\varepsilon} + \boldsymbol{\alpha}\mathrm{E} + \boldsymbol{\zeta}\mathrm{H} + Z_{14}\boldsymbol{\chi}^{\mathbf{D}}\theta - Z_{15}\boldsymbol{\chi}^{\mathbf{D}}\nabla^2\theta + \mathbf{Z_{17}} \tag{73}$$

$$\mathbf{B} = \mathbf{b}\boldsymbol{\varepsilon} + \boldsymbol{\zeta}\mathrm{E} + \boldsymbol{\beta}\mathrm{H} + Z_{14}\boldsymbol{\chi}^{\mathbf{B}}\theta - Z_{15}\boldsymbol{\chi}^{\mathbf{B}}\nabla^2\theta + \mathbf{Z_{18}} \tag{74}$$

where

$$Z_8 = \frac{1}{\Delta t}, \quad \mathbf{Z_9} = \frac{1}{\Delta t}\mathbf{D}^{(n-1)}, \quad \mathbf{Z_{10}} = \frac{1}{\Delta t}\mathbf{A}^{(n-1)}$$

$$Z_{11} = 1 + \frac{1}{\Delta t}w_8\tau_4, \quad Z_{12} = \left(1 + \frac{1}{\Delta t}w_8\tau_4\right)w_9 a$$

$$\mathbf{Z_{19}}_{(0<\alpha\le1)} = -Z_4\frac{1}{\Gamma(\alpha)}\frac{1}{\alpha}\frac{1}{\Delta t^{1-\alpha}}\left[-\nabla\theta^{(n-1)}\right] - Z_5\frac{1}{\Gamma(\alpha)}\frac{1}{\alpha}\frac{1}{\Delta t^{1-\alpha}}\left[-\nabla\dot{\theta}^{(n-1)}\right] - Z_4\frac{1}{\Gamma(\alpha)}\frac{1}{\alpha}\frac{1}{\Delta t^{1-\alpha}}\sum_{i=1}^{n-1}\left[\nabla\theta^{(i)} - \nabla\theta^{(i-1)}\right]\left[(n-i+1)^{\alpha-1} - (n-i)^{\alpha-1}\right]$$
$$- Z_5\frac{1}{\Gamma(\alpha)}\frac{1}{\alpha}\frac{1}{\Delta t^{1-\alpha}}\sum_{i=1}^{n-1}\left(\nabla\dot{\theta}^{(i)} - \nabla\dot{\theta}^{(i-1)}\right)\left[(n-i+1)^{\alpha-1} - (n-i)^{\alpha-1}\right]$$

$$\begin{aligned} Z_{13} = \frac{1}{\Delta t}\Big[&T_0\Delta t\left(\boldsymbol{\chi}^{\boldsymbol{\sigma}}\boldsymbol{\varepsilon}^{(n-1)} + \boldsymbol{\chi}^{\mathbf{D}}\mathbf{E}^{(n-1)} + \boldsymbol{\chi}^{\mathbf{B}}\mathbf{H}^{(n-1)}\right) \\ &+ \rho c_{\mathrm{E}}\Delta t\theta^{(n-1)} - \rho c_{\mathrm{E}}w_9 a\Delta t\nabla^2\theta^{(n-1)} \\ &+ 2w_8\tau_3\rho c_{\mathrm{E}}\theta^{(n-1)} - w_8\tau_3\rho c_{\mathrm{E}}\theta^{(n-2)} \\ &- 2w_8\tau_3\rho c_{\mathrm{E}}w_9 a\nabla^2\theta^{(n-1)} - w_8\tau_3\rho c_{\mathrm{E}}w_9 a\nabla^2\theta^{(n-2)}\Big] \end{aligned}$$

$$Z_{14} = 1 + \frac{1}{\Delta t}w_8\tau_3, \quad Z_{15} = \left(1 + \frac{1}{\Delta t}w_8\tau_3\right)w_9 a$$

$$\mathbf{Z_{16}} = \frac{1}{\Delta t}\left(w_8\tau_3\boldsymbol{\chi}^{\boldsymbol{\sigma}}\theta^{(n-1)} - w_8\tau_3\boldsymbol{\chi}^{\boldsymbol{\sigma}}w_9 a\nabla^2\theta^{(n-1)}\right)$$

$$\mathbf{Z_{17}} = \frac{1}{\Delta t}\left(-w_8\tau_3\boldsymbol{\chi}^{\mathbf{D}}\theta^{(n-1)} + w_8\tau_3\boldsymbol{\chi}^{\mathbf{D}}w_9 a\nabla^2\theta^{(n-1)}\right)$$

$$\mathbf{Z_{18}} = \frac{1}{\Delta t}\left(-w_8\tau_3\boldsymbol{\chi}^{\mathbf{B}}\theta^{(n-1)} + w_8\tau_3\boldsymbol{\chi}^{\mathbf{B}}w_9 a\nabla^2\theta^{(n-1)}\right)$$

When $0 < \beta \le 1$, Caputo definition of time fractional derivative gives:

$$D^{\beta}f(t) = \frac{1}{\Gamma(1-\beta)}\int_0^t \frac{\partial f}{\partial t}(t-\tau)^{-\tau}\mathrm{d}\tau \tag{75}$$

Replacing the integral term in (75) with a summation, and approximating the first-order time derivative by first-order backward difference, one has:

$$\begin{aligned} D^{\beta}f(t) &= \frac{1}{\Gamma(1-\beta)}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{\partial f}{\partial t}(t-\tau)^{-\beta}\mathrm{d}\tau \\ &= \frac{1}{\Gamma(1-\beta)}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{f^{(i)}-f^{(i-1)}}{\Delta t}(t-\tau)^{-\beta}\mathrm{d}\tau \\ &= \frac{1}{\Gamma(1-\beta)}\sum_{i=1}^{n}\frac{f^{(i)}-f^{(i-1)}}{\Delta t}\int_{t_{i-1}}^{t_i}(t-\tau)^{-\beta}\mathrm{d}\tau \\ &= \frac{1}{\Gamma(1-\beta)}\frac{1}{1-\beta}\frac{1}{\Delta t^{\beta}}\sum_{i=1}^{n}\left[f^{(i)}-f^{(i-1)}\right] \\ &\quad\times\left[(n-i+1)^{1-\beta} - (n-i)^{1-\beta}\right] \end{aligned} \tag{76}$$

In view of Eq. (75), Eq. (22) ($0<\alpha\le1$) has the alternative form:

$$\begin{aligned} &w_1\mathbf{q} + (w_2\tau_1 + w_3)\dot{\mathbf{q}} + \frac{Z_4}{\alpha\Gamma(\alpha)\Delta t^{1-\alpha}}\nabla\theta + \frac{Z_5}{\alpha\Gamma(\alpha)\Delta t^{1-\alpha}}\nabla\dot{\theta} \\ &\quad = \mathbf{Z_{19}}_{(0<\alpha\le1)} \end{aligned} \tag{77}$$

where

In view of Eq. (68), Eq. (77) can be further expressed as:

$$Z_{20(0<\alpha\le1)}\mathbf{q} + Z_{21(0<\alpha\le1)}\nabla\theta = \mathbf{Z_{22}}_{(0<\alpha\le1)} \tag{78}$$

where

$$Z_{20(0<\alpha\le1)} = (\Delta t w_1 + w_2\tau_1 + w_3), \quad Z_{21(0<\alpha\le1)} = \frac{\Delta t Z_4 + Z_5}{\Gamma(\alpha)}\frac{1}{\alpha}\frac{1}{\Delta t^{1-\alpha}}$$

$$\mathbf{Z_{22}}_{(0<\alpha\le1)} = \Delta t\mathbf{Z_{19}} + (w_2\tau_1 + w_3)\mathbf{q}^{(n-1)} + \frac{Z_5}{\Gamma(\alpha)}\frac{1}{\alpha}\frac{1}{\Delta t^{1-\alpha}}(\nabla\theta)^{(n-1)}$$

When $1 \le \alpha \le 2$, Eq. (22) may be written:

$$w_1 D^{\alpha-1}\mathbf{q} + (w_2\tau_1 + w_3)D^{\alpha-1}\dot{\mathbf{q}} = -w_4 k\nabla\theta - w_5 k^*\nabla\theta - w_6 k\nabla\dot{\theta} - w_7 k\tau_2\nabla\dot{\theta} \quad (79)$$

Considering Eq. (76), one obtains:

$$\frac{w_1}{\Gamma(2-\alpha)(2-\alpha)t'^{\alpha-1}}\mathbf{q} + \frac{w_2\tau_1 + w_3}{\Gamma(2-\alpha)(2-\alpha)t'^{\alpha-1}}\dot{\mathbf{q}} + Z_4\nabla\theta + Z_5\nabla\dot{\theta} = \mathbf{Z_{19}}_{(1\le\alpha\le 2)} \quad (80)$$

where

$$\begin{aligned}\mathbf{Z_{19}}_{(1\le\alpha\le 2)} = {} & w_1\frac{1}{\Gamma(2-\alpha)}\frac{1}{2-\alpha}\frac{1}{\Delta t^{\alpha-1}}\left[-q^{(n-1)}\right] + (w_2\tau_1+w_3)\frac{1}{\Gamma(2-\alpha)}\frac{1}{2-\alpha}\frac{1}{\Delta t^{\alpha-1}}\left[-\dot{q}^{(n-1)}\right] \\ & - w_1\frac{1}{\Gamma(2-\alpha)}\frac{1}{2-\alpha}\frac{1}{\Delta t^{\alpha-1}}\sum_{i=1}^{n-1}\left[q^{(i)} - q^{(i-1)}\right]\left[(n-i+1)^{\alpha-1} - (n-i)^{\alpha-1}\right] \\ & - (w_2\tau_1+w_3)\frac{1}{\Gamma(2-\alpha)}\frac{1}{2-\alpha}\frac{1}{\Delta t^{\alpha-1}}\sum_{i=1}^{n-1}\left[\dot{q}^{(i)} - \dot{q}^{(i-1)}\right]\left[(n-i+1)^{\alpha-1} - (n-i)^{\alpha-1}\right]\end{aligned}$$

In view of (68), Eq. (80) has the form:

$$Z_{20(1\le\alpha\le 2)}\mathbf{q} + Z_{21(1\le\alpha\le 2)}\nabla\theta = \mathbf{Z_{22}}_{(1\le\alpha\le 2)} \quad (81)$$

where

$$Z_{20(1\le\alpha\le 2)} = \frac{\Delta t w_1 + w_2\tau_1 + w_3}{\Gamma(2-\alpha)(2-\alpha)\Delta t^{\alpha-1}}, \quad Z_{21(1\le\alpha\le 2)} = \Delta t Z_4 + Z_5$$

$$\mathbf{Z_{22}}_{(1\le\alpha\le 2)} = \Delta t\mathbf{Z_{19}} + \frac{w_2\tau_1 + w_3}{\Gamma(2-\alpha)(2-\alpha)\Delta t^{\alpha-1}}\mathbf{q}^{(n-1)} + Z_5(\nabla\theta)^{(n-1)}$$

For convenience, the general form of Eqs. (78) and (81),

$$Z_{20}\mathbf{q} + Z_{21}\nabla\theta = \mathbf{Z_{22}} \quad (82)$$

will be used in the following.

An energy-like trial functional with thirty-six independent variables, i.e., ($\boldsymbol{\sigma}$,$\boldsymbol{\varepsilon}$,$\mathbf{u}$,θ,$\mathbf{q}$,$\mathbf{D}$,$\mathbf{E}$,φ,$\mathbf{B}$,$\mathbf{H}$,$\mathbf{A}$), can be established as:

$$J(\boldsymbol{\sigma},\boldsymbol{\varepsilon},\mathbf{u},\theta,\mathbf{q},\mathbf{D},\mathbf{E},\varphi,\mathbf{B},\mathbf{H},\mathbf{A}) = \int_{t^{(n-1)}}^{t^{(n)}}\int_{v} L\,\mathrm{d}v\,\mathrm{d}t + IB \quad (83)$$

where

$$L = \boldsymbol{\sigma} : \left(\boldsymbol{\varepsilon} - 0.5\nabla\mathbf{u} - 0.5\nabla\mathbf{u}^T\right) + F \quad (84)$$

$$IB = \sum_{i=1}^{8}\int_{t^{(n-1)}}^{t^{(n)}}\int_{s_i} G_i\,\mathrm{d}s\,\mathrm{d}t$$

in which F and $G_i(i = 1,2,\ldots,10)$ are unknown functions to be determined below.

The stationary condition with respect to $\boldsymbol{\sigma}$ in Eq. (83) has the form:

$$\boldsymbol{\varepsilon} - 0.5\nabla\mathbf{u} - 0.5\nabla\mathbf{u}^T + \frac{\delta F}{\delta\boldsymbol{\sigma}} = 0 \quad (85)$$

To satisfy Eq. (18), one has:

$$\frac{\delta F}{\delta\boldsymbol{\sigma}} = 0$$

which indicates that F is not related to $\boldsymbol{\sigma}$ and its derivatives.

The stationary condition for $\boldsymbol{\varepsilon}$ appearing in Eq. (83) can be expressed as:

$$\boldsymbol{\sigma} + \frac{\delta F}{\delta\boldsymbol{\varepsilon}} = 0 \quad (86)$$

In view of Eq. (72), one obtains:

$$F = \boldsymbol{\varepsilon} : \left(-0.5\mathbf{c}\boldsymbol{\varepsilon} + \mathbf{dE} + \mathbf{bH} + Z_{14}\boldsymbol{\chi}^{\boldsymbol{\sigma}}\theta - Z_{15}\boldsymbol{\chi}^{\boldsymbol{\sigma}}\nabla^2\theta - \mathbf{Z_{16}}\right) + F_1 \quad (87)$$

Introducing (87) into (84) leads to:

$$\begin{aligned}L = {} & \boldsymbol{\sigma} : \left(\boldsymbol{\varepsilon} - 0.5\nabla\mathbf{u} - 0.5\nabla\mathbf{u}^T\right) \\ & + \boldsymbol{\varepsilon} : \left(-0.5\mathbf{c}\boldsymbol{\varepsilon} + \mathbf{dE} + \mathbf{bH} + Z_{14}\boldsymbol{\chi}^{\boldsymbol{\sigma}}\theta - Z_{15}\boldsymbol{\chi}^{\boldsymbol{\sigma}}\nabla^2\theta - \mathbf{Z_{16}}\right) + F_1\end{aligned} \quad (88)$$

The trial Euler equation for $\mathbf{u}$ in Eq. (83) can be written as:

$$\nabla\cdot\boldsymbol{\sigma} + \frac{\delta F_1}{\delta\mathbf{u}} = 0 \quad (89)$$

Upon setting:

$$F_1 = \mathbf{f}\cdot\mathbf{u} + 0.5\rho\dot{\mathbf{u}}\cdot\dot{\mathbf{u}} + F_2 \quad (90)$$

one finds that Eq. (89) satisfies Eq. (14). Substitution of (90) into (88) results in:

$$\begin{aligned}L = {} & \boldsymbol{\sigma} : \left(\boldsymbol{\varepsilon} - 0.5\nabla\mathbf{u} - 0.5\nabla\mathbf{u}^T\right) \\ & + \boldsymbol{\varepsilon} : \left(-0.5\mathbf{c}\boldsymbol{\varepsilon} + \mathbf{dE} + \mathbf{bH} + Z_{14}\boldsymbol{\chi}^{\boldsymbol{\sigma}}\theta - Z_{15}\boldsymbol{\chi}^{\boldsymbol{\sigma}}\nabla^2\theta - \mathbf{Z_{16}}\right) \\ & + \mathbf{f}\cdot\mathbf{u} + 0.5\rho\dot{\mathbf{u}}\cdot\dot{\mathbf{u}} + F_2\end{aligned} \quad (91)$$

The stationary condition with respect to θ in (83) is:

$$Z_{14}\boldsymbol{\varepsilon} : \boldsymbol{\chi}^{\boldsymbol{\sigma}} + \frac{\delta F_2}{\delta\theta} = 0$$

Considering Eq. (71), one obtains:

$$F_2 = \frac{\theta}{T_0}\left[Z_{14}\Delta t\nabla\mathbf{q} + T_0 Z_{14}\left(\boldsymbol{\chi}^{\mathbf{D}}\cdot\mathbf{E} + \boldsymbol{\chi}^{\mathbf{B}}\cdot\mathbf{H}\right) - \rho c_E Z_{12}Z_{14}\nabla^2\theta - Z_{13}Z_{14}\right] - \frac{1}{2T_0}\rho c_E Z_{11}Z_{14}\theta^2 + F_3 \quad (92)$$

Combination of (91) and (92) yields:

$$L = \boldsymbol{\sigma} : \left[\boldsymbol{\varepsilon} - 0.5\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right)\right] + \boldsymbol{\varepsilon} : \left(-0.5\mathbf{c}\boldsymbol{\varepsilon} + \mathbf{dE} + \mathbf{bH} + Z_{14}\boldsymbol{\chi}^{\boldsymbol{\sigma}}\theta - Z_{15}\boldsymbol{\chi}^{\boldsymbol{\sigma}}\nabla^2\theta - \mathbf{Z_{16}}\right) + 0.5\rho\dot{\mathbf{u}}\cdot\dot{\mathbf{u}} + \mathbf{f}\cdot\mathbf{u} + \frac{\theta}{T_0}\left[Z_{14}\Delta t\nabla\mathbf{q} + T_0Z_{14}\left(\boldsymbol{\chi}^{\mathbf{D}}\cdot\mathbf{E} + \boldsymbol{\chi}^{\mathbf{B}}\cdot\mathbf{H}\right) - \rho c_{\mathrm{E}}Z_{12}Z_{14}\nabla^2\theta - Z_{13}Z_{14}\right] - \frac{1}{2T_0}\rho c_{\mathrm{E}}Z_{11}Z_{14}\theta^2 + F_3 \quad (93)$$

The trial Euler equation for $\mathbf{q}$ in (83) has the form:

$$-\frac{Z_{14}\Delta t}{T_0}\nabla\theta + \frac{\delta F_3}{\delta\mathbf{q}} = 0$$

In view of Eq. (82), one has:

$$F_3 = \frac{Z_{14}\Delta t\mathbf{q}}{Z_{21}T_0}\cdot\left(-0.5Z_{20}\mathbf{q} + \mathbf{Z_{22}}\right) + F_4 \quad (94)$$

Substitution of (94) into (93) leads to:

$$L = \boldsymbol{\sigma} : \left(\boldsymbol{\varepsilon} - 0.5\nabla\mathbf{u} - 0.5\nabla\mathbf{u}^T\right) + \boldsymbol{\varepsilon} : \left(-0.5\mathbf{c}\boldsymbol{\varepsilon} + \mathbf{dE} + \mathbf{bH} + Z_{14}\boldsymbol{\chi}^{\boldsymbol{\sigma}}\theta - Z_{15}\boldsymbol{\chi}^{\boldsymbol{\sigma}}\nabla^2\theta - \mathbf{Z_{16}}\right) + \mathbf{f}\cdot\mathbf{u} + 0.5\rho\dot{\mathbf{u}}\cdot\dot{\mathbf{u}} + \frac{\theta}{T_0}\left[Z_{14}\Delta t\nabla\mathbf{q} + T_0Z_{14}\left(\boldsymbol{\chi}^{\mathbf{D}}\cdot\mathbf{E} + \boldsymbol{\chi}^{\mathbf{B}}\cdot\mathbf{H}\right) - \rho c_{\mathrm{E}}Z_{12}Z_{14}\nabla^2\theta - Z_{13}Z_{14}\right] - \frac{1}{2T_0}\rho c_{\mathrm{E}}Z_{11}Z_{14}\theta^2 + \frac{Z_{14}\Delta t\mathbf{q}}{Z_{21}T_0}\cdot\left(-0.5Z_{20}\mathbf{q} + \mathbf{Z_{22}}\right) + F_4 \quad (95)$$

The stationary condition for $\mathbf{E}$ can be written as:

$$\boldsymbol{\varepsilon}\mathrm{d} + \theta Z_{14}\boldsymbol{\chi}^{\mathbf{D}} + \frac{\delta F_4}{\delta\mathbf{E}} = 0 \quad (96)$$

Combining (73) and (96) leads to:

$$F_4 = -\mathbf{D}\cdot\mathbf{E} + 0.5\boldsymbol{\alpha}\mathrm{E}\cdot\mathbf{E} + \mathbf{E}\cdot\boldsymbol{\zeta}\mathrm{H} - Z_{15}\mathbf{E}\cdot\boldsymbol{\chi}^{\mathbf{D}}\nabla^2\theta + \mathbf{E}\cdot\mathbf{Z_{17}} + F_5 \quad (97)$$

Introducing (97) into (95), one obtains:

$$L = \boldsymbol{\sigma} : \left(\boldsymbol{\varepsilon} - 0.5\nabla\mathbf{u} - 0.5\nabla\mathbf{u}^T\right) + \boldsymbol{\varepsilon} : \left(-0.5\mathbf{c}\boldsymbol{\varepsilon} + \mathbf{dE} + \mathbf{bH} + Z_{14}\boldsymbol{\chi}^{\boldsymbol{\sigma}}\theta - Z_{15}\boldsymbol{\chi}^{\boldsymbol{\sigma}}\nabla^2\theta - \mathbf{Z_{16}}\right) + \mathbf{f}\cdot\mathbf{u} + 0.5\rho\dot{\mathbf{u}}\cdot\dot{\mathbf{u}} + \frac{\theta}{T_0}\left[Z_{14}\Delta t\nabla\mathbf{q} + T_0Z_{14}\left(\boldsymbol{\chi}^{\mathbf{D}}\cdot\mathbf{E} + \boldsymbol{\chi}^{\mathbf{B}}\cdot\mathbf{H}\right) - \rho c_{\mathrm{E}}Z_{12}Z_{14}\nabla^2\theta - Z_{13}Z_{14}\right] - \frac{1}{2T_0}\rho c_{\mathrm{E}}Z_{11}Z_{14}\theta^2 + \frac{Z_{14}\Delta t\mathbf{q}}{Z_{21}T_0}\cdot\left(-0.5Z_{20}\mathbf{q} + \mathbf{Z_{22}}\right) - \mathbf{D}\cdot\mathbf{E} + 0.5\boldsymbol{\alpha}\mathrm{E}\cdot\mathbf{E} + \mathbf{E}\cdot\boldsymbol{\zeta}\mathrm{H} - Z_{15}\mathbf{E}\cdot\boldsymbol{\chi}^{\mathbf{D}}\nabla^2\theta + \mathbf{E}\cdot\mathbf{Z_{17}} + F_5 \quad (98)$$

The trial Euler equation with respect to $\mathbf{D}$ in (83) can be expressed as:

$$-\mathbf{E} + \frac{\delta F_5}{\delta\mathbf{D}} = 0$$

Together with Eq. (70), it follows that:

$$F_5 = -\mathbf{D}\cdot\nabla\varphi - Z_8\mathbf{D}\cdot\mathbf{A} + \mathbf{D}\cdot\mathbf{Z_{10}} + F_6 \quad (99)$$

Substituting (99) into (98), one has:

$$L = \boldsymbol{\sigma} : \left(\boldsymbol{\varepsilon} - 0.5\nabla\mathbf{u} - 0.5\nabla\mathbf{u}^T\right) + \boldsymbol{\varepsilon} : \left(-0.5\mathbf{c}\boldsymbol{\varepsilon} + \mathbf{dE} + \mathbf{bH} + Z_{14}\boldsymbol{\chi}^{\boldsymbol{\sigma}}\theta - Z_{15}\boldsymbol{\chi}^{\boldsymbol{\sigma}}\nabla^2\theta - \mathbf{Z_{16}}\right) + \mathbf{f}\cdot\mathbf{u} + 0.5\rho\dot{\mathbf{u}}\cdot\dot{\mathbf{u}} + \frac{\theta}{T_0}\left[Z_{14}\Delta t\nabla\mathbf{q} + T_0Z_{14}\left(\boldsymbol{\chi}^{\mathbf{D}}\cdot\mathbf{E} + \boldsymbol{\chi}^{\mathbf{B}}\cdot\mathbf{H}\right) - \rho c_{\mathrm{E}}Z_{12}Z_{14}\nabla^2\theta - Z_{13}Z_{14}\right] - \frac{1}{2T_0}\rho c_{\mathrm{E}}Z_{11}Z_{14}\theta^2 + \frac{Z_{14}\Delta t\mathbf{q}}{Z_{21}T_0}\cdot\left(-0.5Z_{20}\mathbf{q} + \mathbf{Z_{22}}\right) - \mathbf{D}\cdot\mathbf{E} + 0.5\boldsymbol{\alpha}\mathrm{E}\cdot\mathbf{E} + \mathbf{E}\cdot\boldsymbol{\zeta}\mathrm{H} - Z_{15}\mathbf{E}\cdot\boldsymbol{\chi}^{\mathbf{D}}\nabla^2\theta + \mathbf{E}\cdot\mathbf{Z_{17}} - \mathbf{D}\cdot\nabla\varphi - Z_8\mathbf{D}\cdot\mathbf{A} + \mathbf{D}\cdot\mathbf{Z_{10}} + F_6 \quad (100)$$

The stationary condition with respect to φ in (83) has the form:

$$\nabla\cdot\mathbf{D} + \frac{\delta F_6}{\delta\varphi} = 0$$

In view of Eq. (15), one gets:

$$F_6 = -\rho_e\varphi + F_7$$

from which one has:

$$\begin{aligned}L =& \boldsymbol{\sigma} : \left(\boldsymbol{\varepsilon} - 0.5\nabla\mathbf{u} - 0.5\nabla\mathbf{u}^T\right) \\ &+ \boldsymbol{\varepsilon} : \left(-0.5\mathbf{c}\boldsymbol{\varepsilon} + \mathbf{dE} + \mathbf{bH} + Z_{14}\boldsymbol{\chi}^{\boldsymbol{\sigma}}\theta - Z_{15}\boldsymbol{\chi}^{\boldsymbol{\sigma}}\nabla^2\theta - \mathbf{Z_{16}}\right) + \mathbf{f}\cdot\mathbf{u} \\ &+ 0.5\rho\dot{\mathbf{u}}\cdot\dot{\mathbf{u}} + \frac{\theta}{T_0}\Big[Z_{14}\Delta t\nabla\mathbf{q} + T_0 Z_{14}\left(\boldsymbol{\chi}^{\mathbf{D}}\cdot\mathbf{E} + \boldsymbol{\chi}^{\mathbf{B}}\cdot\mathbf{H}\right) \\ &- \rho c_{\mathrm{E}} Z_{12} Z_{14}\nabla^2\theta - Z_{13}Z_{14}\Big] - \frac{1}{2T_0}\rho c_{\mathrm{E}} Z_{11} Z_{14}\theta^2 \\ &+ \frac{Z_{14}\Delta t\mathbf{q}}{Z_{21}T_0}\cdot(-0.5Z_{20}\mathbf{q} + \mathbf{Z_{22}}) - \mathbf{D}\cdot\mathbf{E} + 0.5\alpha\mathrm{E}\cdot\mathbf{E} + \mathbf{E}\cdot\zeta\mathrm{H} \\ &- Z_{15}\mathbf{E}\cdot\boldsymbol{\chi}^{\mathbf{D}}\nabla^2\theta + \mathbf{E}\cdot\mathbf{Z_{17}} - \mathbf{D}\cdot\nabla\varphi - Z_8\mathbf{D}\cdot\mathbf{A} + \mathbf{D}\cdot\mathbf{Z_{10}} - \rho_{\mathrm{e}}\varphi + F_7\end{aligned} \tag{101}$$

The trial Euler equation for **H** in (83) reads:

$$\boldsymbol{\varepsilon}\mathrm{b} + \theta Z_{14}\boldsymbol{\chi}^{\mathbf{B}} + \mathbf{E}\zeta + \frac{\delta F_7}{\delta\mathbf{H}} = 0 \tag{102}$$

Combination of (74) and (102) yields:

$$F_7 = -\mathbf{H}\cdot\mathbf{B} + 0.5\mathbf{H}\cdot\beta\mathrm{H} - Z_{15}\mathbf{H}\cdot\boldsymbol{\chi}^{\mathbf{B}}\nabla^2\theta + \mathbf{H}\cdot\mathbf{Z_{18}} + F_8 \tag{103}$$

Introducing (103) into (101), one has:

$$\begin{aligned}L =& \boldsymbol{\sigma} : \left(\boldsymbol{\varepsilon} - 0.5\nabla\mathbf{u} - 0.5\nabla\mathbf{u}^T\right) \\ &+ \boldsymbol{\varepsilon} : \left(-0.5\mathbf{c}\boldsymbol{\varepsilon} + \mathbf{dE} + \mathbf{bH} + Z_{14}\boldsymbol{\chi}^{\boldsymbol{\sigma}}\theta - Z_{15}\boldsymbol{\chi}^{\boldsymbol{\sigma}}\nabla^2\theta - \mathbf{Z_{16}}\right) + \mathbf{f}\cdot\mathbf{u} \\ &+ 0.5\rho\dot{\mathbf{u}}\cdot\dot{\mathbf{u}} + \frac{\theta}{T_0}\Big[Z_{14}\Delta t\nabla\mathbf{q} + T_0 Z_{14}\left(\boldsymbol{\chi}^{\mathbf{D}}\cdot\mathbf{E} + \boldsymbol{\chi}^{\mathbf{B}}\cdot\mathbf{H}\right) \\ &- \rho c_{\mathrm{E}} Z_{12} Z_{14}\nabla^2\theta - Z_{13}Z_{14}\Big] - \frac{1}{2T_0}\rho c_{\mathrm{E}} Z_{11} Z_{14}\theta^2 \\ &+ \frac{Z_{14}\Delta t\mathbf{q}}{Z_{21}T_0}\cdot(-0.5Z_{20}\mathbf{q} + \mathbf{Z}_{22}) - \mathbf{D}\cdot\mathbf{E} + 0.5\alpha\mathrm{E}\cdot\mathbf{E} + \mathbf{E}\cdot\zeta\mathrm{H} \\ &- Z_{15}\mathbf{E}\cdot\boldsymbol{\chi}^{\mathbf{D}}\nabla^2\theta + \mathbf{E}\cdot\mathbf{Z_{17}} - \mathbf{D}\cdot\nabla\varphi - Z_8\mathbf{D}\cdot\mathbf{A} + \mathbf{D}\cdot\mathbf{Z}_{10} - \rho_{\mathrm{e}}\varphi \\ &- \mathbf{H}\cdot\mathbf{B} + 0.5\mathbf{H}\cdot\beta\mathrm{H} - Z_{15}\mathbf{H}\cdot\boldsymbol{\chi}^{\mathbf{B}}\nabla^2\theta + \mathbf{H}\cdot\mathbf{Z_{18}} + F_8\end{aligned} \tag{104}$$

The stationary condition with respect to **A** can be written as:

$$-Z_8\mathbf{D} + \frac{\delta F_8}{\delta\mathbf{A}} = 0$$

In view of (69), one has:

$$F_8 = (\nabla\times\mathbf{H})\cdot\mathbf{A} + \mathbf{Z_9}\cdot\mathbf{A} - \mathbf{J}\cdot\mathbf{A}$$

from which one gets:

$$\begin{aligned}L =& \boldsymbol{\sigma} : \left(\boldsymbol{\varepsilon} - 0.5\nabla\mathbf{u} - 0.5\nabla\mathbf{u}^T\right) + \boldsymbol{\varepsilon} : \left(-0.5\mathbf{c}\boldsymbol{\varepsilon} + \mathbf{dE} + \mathbf{bH} + Z_{14}\boldsymbol{\chi}^{\boldsymbol{\sigma}}\theta - Z_{15}\boldsymbol{\chi}^{\boldsymbol{\sigma}}\nabla^2\theta - \mathbf{Z_{16}}\right) + \mathbf{f}\cdot\mathbf{u} + 0.5\rho\dot{\mathbf{u}}\cdot\dot{\mathbf{u}} \\ &+ \tfrac{\theta}{T_0}\left[Z_{14}\Delta t\nabla\mathbf{q} + T_0 Z_{14}\left(\boldsymbol{\chi}^{\mathbf{D}}\cdot\mathbf{E} + \boldsymbol{\chi}^{\mathbf{B}}\cdot\mathbf{H}\right) - \rho c_{\mathrm{E}} Z_{12}Z_{14}\nabla^2\theta - Z_{13}Z_{14}\right] - \tfrac{1}{2T_0}\rho c_{\mathrm{E}} Z_{11}Z_{14}\theta^2 + \tfrac{Z_{14}\Delta t\mathbf{q}}{Z_{21}T_0}\cdot(-0.5Z_{20}\mathbf{q} + \mathbf{Z_{22}}) - \mathbf{D}\cdot\mathbf{E} + 0.5\alpha\mathrm{E}\cdot\mathbf{E} \\ &+ \mathbf{E}\cdot\zeta\mathrm{H} - Z_{15}\mathbf{E}\cdot\boldsymbol{\chi}^{\mathbf{D}}\nabla^2\theta + \mathbf{E}\cdot\mathbf{Z_{17}} - \mathbf{D}\cdot\nabla\varphi - Z_8\mathbf{D}\cdot\mathbf{A} + \mathbf{D}\cdot\mathbf{Z_{10}} - \rho_{\mathrm{e}}\varphi - \mathbf{H}\cdot\mathbf{B} + 0.5\mathbf{H}\cdot\beta\mathrm{H} - Z_{15}\mathbf{H}\cdot\boldsymbol{\chi}^{\mathbf{B}}\nabla^2\theta + \mathbf{H}\cdot\mathbf{Z_{18}}(\nabla\times\mathbf{H})\cdot\mathbf{A} + \mathbf{Z_9}\cdot\mathbf{A} - \mathbf{J}\cdot\mathbf{A}\end{aligned} \tag{105}$$

Applying Green's theory on the boundary, one has:

$$\begin{aligned}&\delta\mathbf{u} : -\boldsymbol{\sigma}\cdot\mathbf{n} + \tfrac{\partial G_i}{\partial\mathbf{u}} = 0 \\ &\delta\varphi : -\mathbf{D}\cdot\mathbf{n} + \tfrac{\partial G_i}{\partial\varphi} = 0 \\ &\delta\boldsymbol{\sigma} : \tfrac{\partial G_i}{\partial\boldsymbol{\sigma}} = 0 \\ &\delta\mathbf{D} : \tfrac{\partial G_i}{\partial\mathbf{D}} = 0 \\ &\delta\mathbf{A} : \tfrac{\partial G_i}{\partial\mathbf{A}} = 0 \qquad (i = 1, 2, \ldots, 8) \\ &\delta\mathbf{H} : -\mathbf{A}(\nabla\times\mathbf{I}) + \tfrac{\partial G_i}{\partial\mathbf{H}} = 0 \\ &\delta\theta : \tfrac{\partial G_i}{\partial\theta} = 0 \\ &\delta\mathbf{q} : -\tfrac{Z_{14}Z_{20}t'}{Z_{21}T_0}\theta\mathbf{n} + \tfrac{\partial G_i}{\partial\mathbf{q}} = 0\end{aligned} \tag{106}$$

Considering the boundary equations on $s_i(i = 1,2,\ldots,8)$, one obtains from (106) that:

$$\begin{aligned}&G_1 = \boldsymbol{\sigma}\cdot\mathbf{n}(\mathbf{u} - \overline{\mathbf{u}}), \quad G_2 = \mathbf{F}\cdot\mathbf{u}, \quad G_3 = \mathbf{D}\cdot\mathbf{n}(\varphi - \overline{\varphi}) \\ &G_4 = \overline{d}\varphi, \quad G_5 = \mathbf{A}\left(\nabla\times\mathbf{H} - \overline{\mathbf{h}}\right), \quad G_6 = \overline{\mathbf{A}}(\nabla\times\mathbf{H}) \\ &G_7 = \tfrac{Z_{14}Z_{20}\Delta t}{Z_{21}T_0}\theta(\mathbf{q}\cdot\mathbf{n} - \overline{q}), \quad G_8 = \tfrac{Z_{14}Z_{20}\cdot t}{Z_{21}T_0}\mathbf{q}\cdot\mathbf{n}\overline{\theta}\end{aligned} \tag{107}$$

Finally, substitution of (105), (107) into (83) leads to the generalized variational principle of FOGEMTE.

Note that several specialized variational theorems can be obtained from the present generalized variational principle by introducing suitable constraints. For typical example, if Eq. (18) is taken as the constraint, Eq. (105) simplifies to:

$$\begin{aligned}L =& \boldsymbol{\varepsilon} : \left(-0.5\mathbf{c}\boldsymbol{\varepsilon} + \mathbf{dE} + \mathbf{bH} + Z_{14}\boldsymbol{\chi}^{\boldsymbol{\sigma}}\theta - Z_{15}\boldsymbol{\chi}^{\boldsymbol{\sigma}}\nabla^2\theta - \mathbf{Z_{16}}\right) \\ &+ \mathbf{f}\cdot\mathbf{u} + 0.5\rho\dot{\mathbf{u}}\cdot\dot{\mathbf{u}} + \frac{\theta}{T_0}\Big[Z_{14}\Delta t\nabla\mathbf{q} + T_0 Z_{14}\left(\boldsymbol{\chi}^{\mathbf{D}}\cdot\mathbf{E} + \boldsymbol{\chi}^{\mathbf{B}}\cdot\mathbf{H}\right) \\ &- \rho c_{\mathrm{E}} Z_{12}Z_{14}\nabla^2\theta - Z_{13}Z_{14}\Big] - \frac{1}{2T_0}\rho c_{\mathrm{E}} Z_{11}Z_{14}\theta^2 \\ &+ \frac{Z_{14}\Delta t\mathbf{q}}{Z_{21}T_0}\cdot(-0.5Z_{20}\mathbf{q} + \mathbf{Z_{22}}) - \mathbf{D}\cdot\mathbf{E} + 0.5\alpha\mathrm{E}\cdot\mathbf{E} + \mathbf{E}\cdot\zeta\mathrm{H} \\ &- Z_{15}\mathbf{E}\cdot\boldsymbol{\chi}^{\mathbf{D}}\nabla^2\theta + \mathbf{E}\cdot\mathbf{Z_{17}} - \mathbf{D}\cdot\nabla\varphi - Z_8\mathbf{D}\cdot\mathbf{A} + \mathbf{D}\cdot\mathbf{Z_{10}} - \rho_{\mathrm{e}}\varphi \\ &- \mathbf{H}\cdot\mathbf{B} + 0.5\mathbf{H}\cdot\beta\mathrm{H} - Z_{15}\mathbf{H}\cdot\boldsymbol{\chi}^{\mathbf{B}}\nabla^2\theta + \mathbf{H}\cdot\mathbf{Z_{18}}(\nabla\times\mathbf{H})\cdot\mathbf{A} \\ &+ \mathbf{Z_9}\cdot\mathbf{A} - \mathbf{J}\cdot\mathbf{A}\end{aligned} \tag{108}$$

Then, Equations (83), (107), and (108) represent the newly degenerated variational theorem.

6. Numerical examples

In the previous sections, a unified form of FOGEMTE is introduced, and it may be reduced into FOGTE when the electromagnetic effect is neglected, while the FOGTE can be considered as a summary of reported fractional theories based on ETE, TEWOED, TEWED, and DPL (Youssef, 2010a, 2012; Youssef and Al-Lehaibi, 2010; Abouelregal, 2011; Sur and Kanoria, 2012; Ezzat et al., 2012). In addition, to give a general view of FOGEMTE and provide a further understanding of the problem, the corresponding variational principles are proposed in both variational integral method and semi-inverse method. It is noted that FOGTE based on TRDTE is never reported and is introduced in this contribution for the first time.

In this section, our focuses are placed on the effect of fractional order α on the distribution of the response when the material (thermo-elastic or electro-thermo-elastic) is imposed a sudden heating. In the numerical simulations, the Laplace transformation is implemented, and the adopted model is based on TRDTE. The results are presented graphically.

6.1. A slim strip problem of thermo-elastic material

The equations of FOGTE based on TRDTE are restated as following:

$$\sigma_{ji,j} = \rho\ddot{u}_i, \quad q_{i,i} = -\rho T_0\dot{\eta} \tag{109}$$

$$\sigma_{ij} = c_{ijkl}\varepsilon_{kl} - \chi^{\sigma}_{ij}\left(\theta + \tau_3\dot{\theta}\right) \tag{110}$$

$$\rho\eta = \chi^{\sigma}_{ij}\varepsilon_{ij} + \frac{\rho c_{\mathrm{E}}}{T_0}\left(\theta + \tau_4\dot{\theta}\right) \tag{111}$$

$$q_i = -I^{\alpha-1}k\nabla\theta, \quad \varepsilon_{ij} = 0.5(u_{i,j} + u_{j,i}) \tag{112}$$

Considering a one-dimensional problem of an isotropic thermo-elastic material, the constitutive Equations (110) and (111) are simplified, as:

$$\sigma_x = (\lambda + 2\mu)\frac{\partial u}{\partial x} - \gamma\left(\theta + \tau_3\dot{\theta}\right) \tag{113}$$

$$\rho\eta = \gamma\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\rho c_{\mathrm{E}}}{T_0}\left(\theta + \tau_4\dot{\theta}\right) \tag{114}$$

where λ and μ are Lame's constants, λ=$(3\lambda + 2\mu)\alpha_t$ and α_t is the linear thermal expansion coefficient, $u = u(x,t)$ is the one-dimensional displacement. Then, one obtains from Eqs. (109), (112)–(114):

$$(\lambda + 2\mu)\frac{\mathrm{d}^2u}{\mathrm{d}x^2} - \gamma\left(\frac{\mathrm{d}\theta}{\mathrm{d}x} + \tau_3\frac{\mathrm{d}^2\theta}{\mathrm{d}x\mathrm{d}t}\right) = \rho\frac{\mathrm{d}^2u}{\mathrm{d}t^2} \tag{115}$$

$$\gamma T_0\frac{\mathrm{d}^2u}{\mathrm{d}x\mathrm{d}t} + \rho c_{\mathrm{E}}\left(\frac{\mathrm{d}\theta}{\mathrm{d}t} + \tau_4\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}\right) = I^{\alpha-1}k\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} \tag{116}$$

For convenience, the following dimensionless quantities are introduced:

$$(\tilde{x},\tilde{u},\tilde{\varphi}) = n_1n_2(x,u,\varphi), \ (\tilde{t},\tilde{\tau}_3,\tilde{\tau}_4) = n_1^2n_2(t,\tau_3,\tau_4), \ \tilde{\sigma}_{ij} = \frac{\sigma_{ij}}{\mu}, \ \tilde{\theta} = \frac{\theta}{T_0} \tag{117}$$

where $n_1 = \sqrt{(\lambda + 2\mu)/\rho}$, $n_2 = (\rho c_{\mathrm{E}})/k$. Eqs. (115), (116) and (113) may be rewritten, as:

Table 3
Material constants needed in numerical simulation (copper).

λ	77.6 GPa	α_t	1.78e − 5 m/K	c_{E}	381 J/(kg K)	τ_3	0.05
μ	38.6 GPa	ρ	8945 kg/m^3	T_0	293 K	τ_4	0.05

$$\beta^2\frac{\mathrm{d}^2u}{\mathrm{d}x^2} - b\left(\frac{\mathrm{d}\theta}{\mathrm{d}x} + \tau_3\frac{\mathrm{d}^2\theta}{\mathrm{d}x\mathrm{d}t}\right) = \beta^2\frac{\mathrm{d}^2u}{\mathrm{d}t^2} \tag{118}$$

$$g\frac{\mathrm{d}^2u}{\mathrm{d}x\mathrm{d}t} + \frac{\mathrm{d}\theta}{\mathrm{d}t} + \tau_4\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = I^{\alpha-1}\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} \tag{119}$$

$$\sigma_x = \beta^2\frac{\partial u}{\partial x} - b\left(\theta + \tau_3\dot{\theta}\right) \tag{120}$$

in which, $\beta^2 = (\lambda + 2\mu)/\mu$, $b = \gamma T_0/\mu$, $g = \gamma/kn_2$, and the caps of dimensionless quantities have been left out for brevity. The initial and boundary conditions are given, as:

$$\begin{aligned} & u(x,t) = \dot{u}(x,t) = 0 \quad \text{at } t = 0 \\ & \theta(x,t) = \dot{\theta}(x,t) = 0 \quad \text{at } t = 0 \\ & \theta(x,t) = H(t) \quad \text{at } x = 0 \\ & \sigma(x,t) = 0 \quad \text{at } x = 0 \\ & u(x,t) \to 0 \quad \text{as } x \to \infty, \ t > 0 \\ & \theta(x,t) \to 0 \quad \text{as } x \to \infty, \ t > 0 \end{aligned} \tag{121}$$

which means the strip is prescribed a quiescent initial state with its boundary $x = 0$ free of stress and subjected to a sudden heating. $H(t)$ is the Heaviside unit step function. The copper material is chosen for numerical evaluation, and the constants are shown in Table 3.

Fig. 1 shows the time history of the temperatures at the point $x = 0.1$. When $\alpha = 1$ the FOGTE is reduced to TRDTE, the results are compared with that from Tian et al. (2006), and a perfect coincidence is obtained. It is found that the smaller the fractional order, the larger the velocity of the thermal signal. That is to say, the temperatures of the point rise earlier when the fractional order is small, as shown in Fig. 1. In addition, a small fractional order smoothes the curve of time history of the temperatures (the jump as shown when $\alpha = 1$ is eliminated), and for a longer time the temperature distributions coincide under different fractional order.

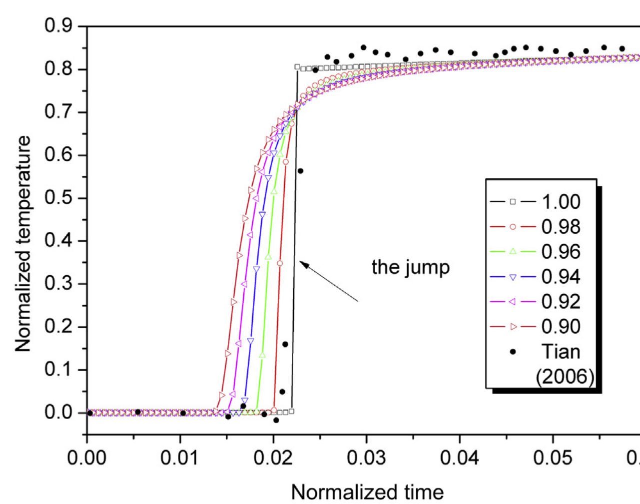

Fig. 1. Time history of the normalized temperature at $x = 0.1$ under different α.

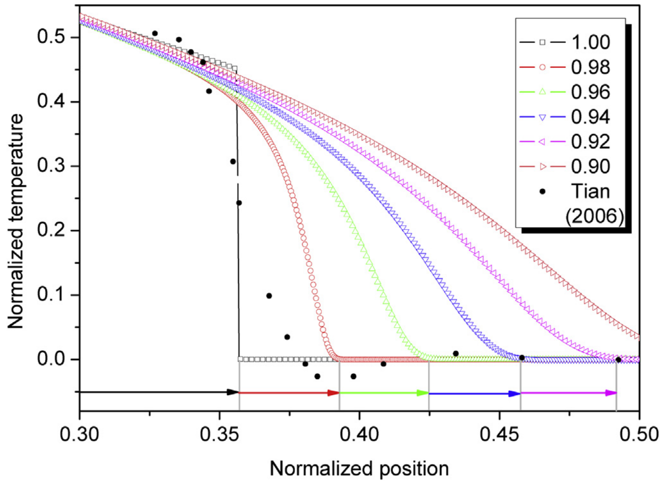

Fig. 2. The distributions of the normalized temperature along x at $t = 0.08$ under different α.

Fig. 2 shows the distributions of temperature along x at $t = 0.08$ under different α. As shown in Fig. 2, TRDTE ($\alpha = 1$) results in a jump at $x = 0.358$ ($x = 1/\sqrt{\tau_4} \times t$), which is called the wave front of thermal wave, and the results are better than that from Tian et al. (2006) as the front of thermal wave is more apparent. It is also found that the smaller the fractional order α, the farthest the thermal signals arrive at (which is indicated by the arrows depicted in Fig. 2.), which agrees with the conclusion made from Fig. 1. In addition, variations of the temperatures for a small α are smoother than for a lager one.

Fig. 3 is the distributions of displacement along x at $t = 0.08$ under different α. From Fig. 3, it is concluded that the displacements have great changes at points A and B. Point A is the position the elastic wave arrives at ($x = \sqrt{(\beta^2/\beta^2)}*t = 0.08$), while the second one (B) depicts the wave front of thermal wave. More importantly, the fractional order can smooth the lager variation at B, but has no effect on the first one (at A). This is rational because the fractional calculus is introduced into the governing equation of the thermal field, see Eq. (119). In addition, we find that the fractional order may reduce the displacement slightly.

To distinguish the displacement induced by elastic and thermal field for FOGTE based on TRDTE ($\alpha = 1$), the lagging time τ_4 is set as 1/4, 4/9, 16/9, 4 to obtain the thermal wave propagating faster or

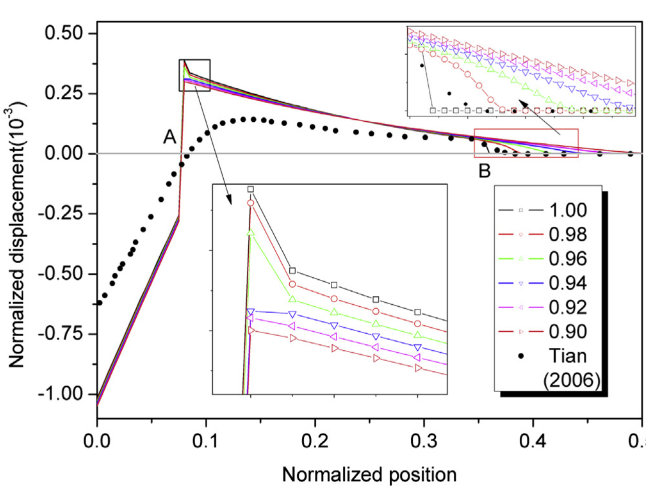

Fig. 3. The distributions of the normalized displacement along x at $t = 0.08$ under different α.

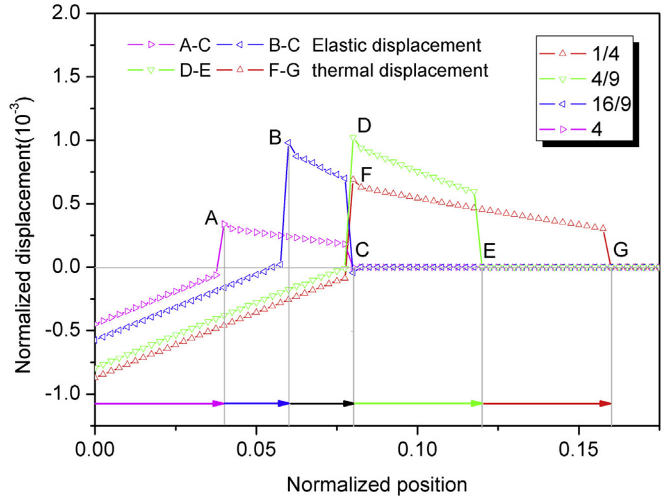

Fig. 4. The distributions of the normalized displacement along x at $t = 0.08$ for different τ_4 ($\alpha = 1$).

slower than the elastic wave, the corresponding velocities are 2, 1.5, 0.75, 0.5 (see Eq. (119)), respectively. While the velocity of elastic wave is always 1 obtained from Eq. (118). The numerical results are presented in Fig. 4.

In Fig. 4 the front of elastic wave is depicted by the black arrow, while the fronts of thermal wave for different τ_4 are shown by the corresponding coloured arrows (in the web version). For $\tau_4 = 16/9$ (4), because the thermal waves arrive at points B (A), it is concluded that the displacement between B (A) and C is induced by the elastic field. In addition, the displacement induced by elastic wave is always 0 after the point C, as a result, the displacement between C and D (E) is contributed by the thermal wave for $\tau_4 = 4/9$ (1/4). In addition, the great changes at point C are the result of elastic wave because it is the front of elastic wave depicted by black arrow, while the lager variations at point A, B, D and E, where the front of thermal wave arrives at, comes from the thermal field.

6.2. A slim strip problem of electro-thermo-elastic material

In this subsection, we consider the response of electro-thermo-elastic material, when subjected to a sudden heating. The equations for one-dimensional case, read:

$$\begin{aligned}
&\sigma_{x,x} = \rho\ddot{u}, \quad q_{x,x} = -\rho T_0\dot{\eta}, \quad D_{x,x} = 0 \\
&\sigma_x = c\frac{du}{dx} - dE - \gamma\left(\theta + \tau_3\dot{\theta}\right), \quad \rho\eta = \gamma\frac{du}{dx} + eE + \frac{\rho c_E}{T_0}\left(\theta + \tau_4\dot{\theta}\right) \\
&D_x = d\frac{du}{dx} + pE + e\left(\theta + \tau_3\dot{\theta}\right), \quad E_x = -\varphi_{,x}, \quad q_x = -I^{\alpha-1}k\theta_{,x}, \\
&\varepsilon_x = u_{,x}
\end{aligned} \tag{122}$$

where c, d, p and e are elastic, piezoelectric, dielectric and pyroelectric constants. Considering the following dimensionless quantities $(\tilde{x}, \tilde{u}, \tilde{\varphi}) = n_1n_2(x, u, \varphi)$, $(\tilde{t}, \tilde{\tau}_3, \tilde{\tau}_4) = n_1^2n_2(t, \tau_3, \tau_4)$, $\tilde{\sigma}_{ij} = \sigma_{ij}/\mu$, $\tilde{\theta} = \theta/T_0$, the governing equations may be rewritten as (the caps are dismissed for brevity):

Table 4
Material constants needed in numerical simulation (cadmium selenide).

c	83.6e9 Pa	γ	0.551e6 N K^{-1} m^{-2}	e	$e = -2.94e-6$ C K^{-1} m^{-2}
d	0.347 Cm^{-2}	ρ	7600 kg/m^3	p	$p = 90.3e-12$ C^2 N^{-1} m^{-2}
c_E	420 J/(kg K)	τ_3	0.05	μ	30.6e9 Pa
T_0	293 K	τ_4	0.05		

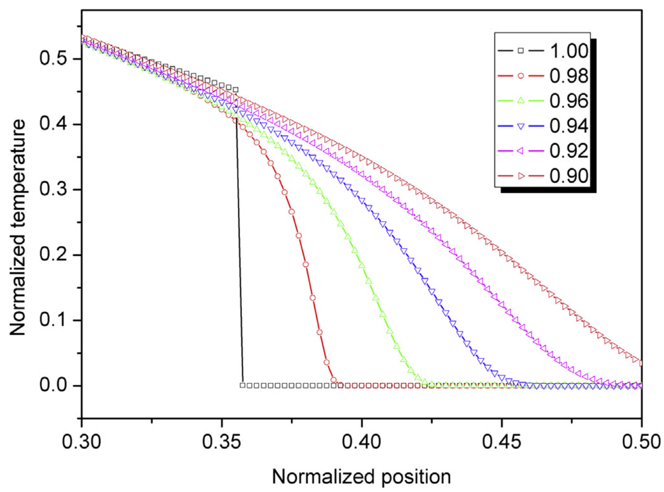

Fig. 5. The distributions of the normalized temperature along x at $t = 0.08$ under different α.

$$\beta^2\frac{d^2u}{dx^2} + \xi\frac{d^2\varphi}{dx^2} - b\left(\frac{d\theta}{dx} + \tau_3\frac{d^2\theta}{dxdt}\right) = \beta^2\frac{d^2u}{dt^2}$$
$$d\frac{d^2u}{dx^2} - p\frac{d^2\varphi}{dx^2} + eT_0\left(\frac{d\theta}{dx} + \tau_3\frac{d^2\theta}{dxdt}\right) = 0 \quad (123)$$
$$g\frac{d^2u}{dxdt} - h\frac{d^2\varphi}{dxdt} + \frac{d\theta}{dt} + \tau_4\frac{d^2\theta}{dt^2} = I^{\alpha-1}\frac{d^2\theta}{dx^2}$$

where $\beta^2 = (\lambda+2\mu)/\mu$, $\xi = d/\mu$, $b = \gamma T_0/\mu$, $g = \gamma/kn_2$, $h = e/\rho c_E$, $n_1 = \sqrt{(\lambda+2\mu)/\rho}$, $n_2 = \rho c_E/k$. The numerical simulation is devoted to cadmium selenide material (see Table 4). The results are presented in following.

As shown in Figs. 5 and 6, the distributions of the temperatures and the displacements are same as that obtained for the thermoelastic material, and the fractional order also has alike effect on them. Fig. 7 is presented to illustrate the induced electrical potential when the material is imposed a sudden heating, and comes to the conclusion that there are also two changes at the front of elastic and thermal wave, and that the fractional order may increase the electrical potential. In addition, the fractional order smoothes the curve of potential around the front of thermal wave, but it has little on the first changes around the front of elastic wave.

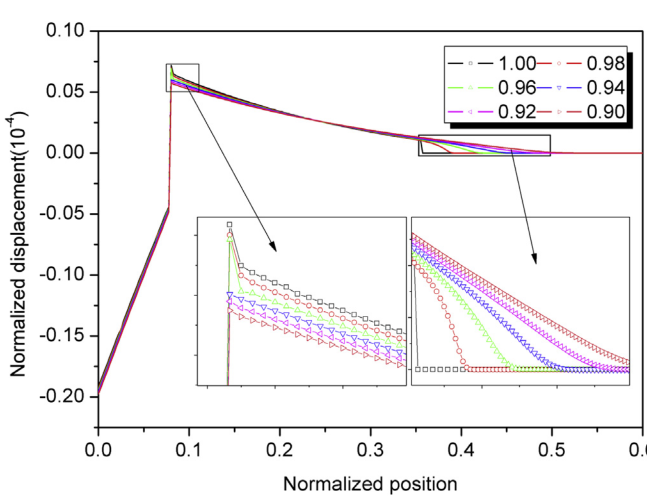

Fig. 6. The distributions of the normalized displacement along x at $t = 0.08$ under different α.

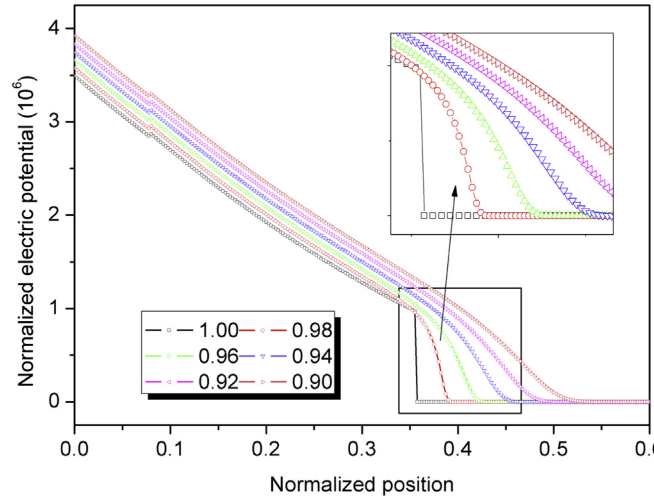

Fig. 7. The distributions of the normalized electric potential along x at $t = 0.08$ under different α.

7. Conclusion

A fractional order generalized electro-magneto-thermo-elasticity (FOGEMTE) theory is developed for anisotropic and linearly electro-magneto-thermo-elastic media by introducing the dynamic electro-magnetic fields. To further generalize the newly developed theory, the general form of several generalized thermoelastic theories, such as the extended thermoelasticity (ETE), temperature rate dependent thermoelasticity (TRDTE), thermoelasticity without energy dissipation (TEWOED), thermoelasticity with energy dissipation (TEWED), and dual-phase-lag thermoelasticity (DPLTE), is introduced. The variational principle of the FOGEMTE theory is formulated using the variational integral method, which can be degenerated to several existing variational theorems. A generalized variational theorem for the FOGEMTE theory is developed with the semi-inverse method. Finally, two numerical examples are given, and the numerical results shows that the fractional order has great effect on the response, and, importantly, it may smooth the lager changes for the response when material is imposed a sudden heating, which is commonly existed in modern engineering. Considering the great effect, fractional theories may have a wide application in future engineering practice, such as: low temperature regimes, amorphous media, colloids, glassy and porous materials, man-made and biological materials or polymers, transient loading.

Acknowledgements

This study was supported by National Science Foundation of China (11172230), the National Basic Research Program of China (2011CB610305), Specialized Research Fund for the Doctoral Program of Higher Education of China (20110201110062) and the National 111 Project of China (B06024).

References

Abouelregal, A.E., 2011. Fractional order generalized thermo-piezoelectric semi-infinite medium with temperature-dependent properties subjected to a ramp-type heating. J. Therm. Stress. 34, 1139–1155.
Biot, M., 1956. Thermoelasticity and irreversible thermo-dynamics. J. Appl. Phys. 27, 240–253.
Bai, Z.B., Lu, H.S., 2005. Positive solutions for boundary value problem of nonlinear fractional differential equation. J. Math. Anal. Appl. 311, 495–505.
Cattaneo, C., 1958. A form of heat equation which eliminates the paradox of instantaneous propagation. C. R. Acad. Sci. 247, 431–433.
Chandrasekharaiah, D.S., 1998. Hyperbolic thermoelasticity: a review of recent literature. Appl. Mech. Rev. 51, 705–729.

Chen, W.Q., Lee, K.Y., Ding, H.J., 2004. General solution for transversely isotropic magneto-electro-thermo-elasticity and the potential theory method. Int. J. Eng. Sci. 42, 1361–1379.
Chen, P., Shen, Y.P., Tian, X.G., 2006. Dynamic potentials and Green's functions of a quasi-plane magneto-electro-elastic medium with inclusion. Int. J. Eng. Sci. 44, 540–553.
Diethelm, K., Ford, N.J., Freed, A.D., Yu, Luchko, 2005. Algorithms for the fractional calculus: a selection of numerical methods. Comput. Methods Appl. Mech. Eng. 194, 743–773.
Deng, J.Q., Ma, L.F., 2010. Existence and uniqueness of solutions of initial value problems for nonlinear fractional differential equations. Appl. Math. Lett. 23, 676–680.
Ezzat, M.A., Karamany, A.S. El, 2011a. Fractional order heat conduction law in magneto-thermoelasticity involving two temperatures. Z. Angew. Math. Phys. 62, 937–952.
Ezzat, M.A., Karamany, A.S. El, 2011b. Fractional order theory of a perfect conducting thermoelastic medium. Can. J. Phys. 89, 311–318.
Ezzat, M.A., Karamany, A.S. El, 2011c. Theory of fractional order in electro-thermoelasticity. Eur. J. Mech. A Solid 30, 491–500.
Ezzat, M.A., 2011. Magneto-thermoelasticity with thermoelectric properties and fractional derivative heat transfer. Physica B 406, 30–35.
Ezzat, M.A., Karamany, A.S. El, Ezzat, S.M., 2012. Two temperature theory in magneto-thermoelasticity with fractional order dual-phase-lag heat transfer. Nucl. Eng. Des. 252, 267–277.
Green, A.E., Lindsay, K., 1972. Thermoelasticity. J. Elast. 2, 1–7.
Green, A.E., Naghdi, P.M., 1977. On thermodynamics and the nature of the second law. Proc. R. Soc. Lond. A 357, 253–270.
Green, A.E., Naghdi, P.M., 1991. A reexamination of the basic results of thermomechanics. Proc. R. Soc. Lond. A 432, 171–194.
Green, A.E., Naghdi, P.M., 1992. On undamped heat waves in an elastic solid. J. Therm. Stress. 15, 252–264.
Green, A.E., Naghdi, P.M., 1993. Thermoelasticity without energy dissipation. J. Elast. 31, 189–208.
Gurtin, M.E., 1964. Variational principles for linear elastodynamics. Arch. Ration. Mech. Anal. 16 (1), 34–50.
Hetnarski, R.B., Ignaczak, J., 1993. Generalized thermoelasticity: closed form solutions. J. Therm. Stress. 16, 473–498.
Hetnarski, R.B., Ignaczak, J., 1994. Generalized thermoelasticity: response of semi-space to a short laser pulse. J. Therm. Stress. 17, 377–396.
He, J.H., 1997. Semi-inverse method of establishing generalized variational principles for fluid mechanics with emphasis on turbomachinery aerodynamics. Int. J. Turbo Jet Eng. 14, 23–28.
He, J.H., 2001. Variational theory for linear magneto-electro-elasticity. Int. J. Nonlin. Sci. Numer. 2, 309–316.
He, J.H., 2002. Generalized variational principles for thermopiezoelectricity. Arch. Appl. Mech. 72, 248–256.
Ignaczak, I., Ostoja-Starzewski, M., 2010. Thermoelasticity with Finite Wave Speeds. Oxford University.
Joseph, D.D., Preziosi, L., 1989. Heat waves. Rev. Mod. Phys. 61, 41–73.
Lord, H., Shulman, Y., 1967. A generalized dynamic theory of thermoelasticity. J. Mech. Phys. Solids 15, 299.
Liang, L.F., Liu, D.K., Song, H.Y., 2005. The generalized quasi-variational principles of non-conservative systems with two kinds of variables. Sci. China Ser. G 48, 600–613.
Mainardi, F., Gorenflo, R., 2000. On mittag-leffler-type functions in fractional evolution processes. J. Comput. Appl. Math. 118, 283–299.
Mallik, S.H., Kanoria, M., 2008. A two dimensional problem for a transversely isotropic generalized thermoelastic thick plate with spatially varying heat source. Eur. J. Mech. A Solid 27, 607–621.
Naotak, N., Hetnarski, R., Tanigawa, Y., 2003. Thermal Stresses, second ed. Taylor & Francis, New York.
Niraula, O.P., Noda, N., 2010. Derivation of material constants in non-linear electro-magneto-thermo-elasticity. J. Therm. Stress. 33, 1011–1034.
Peshkov, V., 1944. Second sound in helium II. J. Phys. 8, 381–382.
Podlubny, I., 1999. Fractional Differential Equations. Academic, New York.
Roychoudhuri, S.K., 2007. On a thermoelastic three-phase-lag model. J. Therm. Stress. 30, 231–238.
Sur, A., Kanoria, M., 2012. Fractional order two-temperature thermoelasticity with finite wave speed. Acta Mech. 223, 2685–2701.
Su, X.W., 2009. Boundary value problem for a coupled system of nonlinear fractional differential equations. Appl. Math. Lett. 22, 64–69.
Sherief, H.H., El-Sayed, A.M.A., Abd, El-Latief A.M., 2010. Fractional order theory of thermoelasticity. Int. J. Solids Struct. 47, 269–275.
Tzou, D.Y., 1995. A unified field approach for heat conduction from macro to micro scales. ASME J. Heat Trans. 117, 8–16.
Taheri, H., Fariborz, S.J., Eslami, M.R., 2005. Thermoelastic analysis of an annulus using the Green-Naghdi model. J. Therm. Stress. 28, 911–927.
Tian, X.G., Shen, Y.P., Chen, C.Q., He, T.H., 2006. A direct finite element method study of generalized thermoelastic problems. Int. J. Solids Struct. 43, 2050–2063.
Tian, X.G., Zhang, J., Shen, Y.P., Lu, T.J., 2007. Finite element method for generalized piezothermoelastic problems. Int. J. Solids Struct. 44, 6330–6339.
Vernotte, P., 1958a. Paradoxes in the continuous theory of the heat conduction. C. R. Acad. Sci. 246, 3154–3155.
Vernotte, P., 1958b. The true heat conduction. C. R. Acad. Sci. 247, 2103.
Vernotte, P., 1961. Some possible complications in the phenomena of thermal conduction. C. R. Acad. Sci. 252, 2190.
Wang, X.Z., Zhou, Y.H., Zheng, X.J., 2002. A generalized variational model of magneto-thermo-elasticity for nonlinearly magnetized ferroelastic bodies. Int. J. Eng. Sci. 40, 1957–1973.
Wang, Z.J., Zheng, D.Z., Zheng, C.B., 2010. Simplified Gurtin-type generalized variational principles for fully dynamic magneto-electro-elasticity with geometrical nonlinearity. Int. J. Solids Struct. 47, 3115–3120.
Youssef, H.M., 2006. Theory of two-temperature generalized thermoelasticity. IMA J. Appl. Math. 71, 1–8.
Youssef, H.M., 2010a. Theory of fractional order generalized thermoelasticity. ASME J. Heat Trans. 132, 6.
Youssef, H.M., 2010b. Variational principle of fractional order generalized thermoelasticity. Appl. Math. Lett. 23, 1183–1187.
Youssef, H.M., 2012. Two-dimensional thermal shock problem of fractional order generalized thermoelasticity. Acta Mech. 223, 1219–1231.
Youssef, H.M., Al-Lehaibi, E.A., 2010. Fractional order generalized thermoelastic half-space subjected to ramp-type heating. Mech. Res. Commun. 37, 448–452.